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Studies in Epistemology, Logic, Methodology,  
and Philosophy of Science

Frédéric Patras

# Contemporary Mathematical Thinking



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# Synthese Library

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and Philosophy of Science

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Frédéric Patras

# Contemporary Mathematical Thinking



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# Preface to the English Translation

This book was written and published, in its original French version, about 20 years ago. One might therefore think that its very title, and especially the adjective “contemporary” should be amended. However, the observation that was its starting point remains valid: that of the end of the structuralist paradigm, to which no other has succeeded, and of the need to rethink mathematics in the light of new practices, new theoretical fields, new difficulties to be faced.

This book was one of the very first to deal with themes that have now become central to the history and philosophy of mathematics: in addition to mathematical structuralism, ideas such as style, the role of the Bourbaki group, its history, its dynamics, its evolution, and the originality of Grothendieckian thought. Nevertheless, available only in French and in its Italian translation, although reissued in both languages, it could not reach many of its potential readers due to the language barrier. I therefore particularly welcome the opportunity given by Springer to publish this translation.

In addition to the fact that the themes that organise its content are currently alive, the book retains an interest and originality that goes beyond its historical positioning. Contrary to most books and works published in the field, it is deliberately aimed at mathematicians who wish to understand the evolution of their discipline during the twentieth century, the evolution of its standards of writing and thought, as well as at philosophers of science and, more generally, all those interested in mathematical thought. It is written by a mathematician by profession who has followed the evolution of his discipline from the inside since the mid-1980s and who is very close to mathematicians such as Pierre Cartier, one of the *de facto* contributors to the construction of the structuralist paradigm from the 1950s onwards, and then one of the first to analyse the need to go beyond it. The analyses made, based on first-hand experience and practice, are therefore necessarily different from those made by a philosopher, whose interests will naturally be directed towards other issues—often logical or ontological—or a historian by profession.

If I were to rewrite this book today, I would mostly develop further the following point. My generation grew up, in France at least, in the spirit of the reform of modern mathematics, which was then casting its very last fires. It was still deeply

influenced by structuralism—a very conceptual, very systematic, very organised way of doing mathematics—and could feel the need for a renewal that would go hand in hand with changes in theoretical interests and changes in practice. The very recent period seems to have gone a little too far in this direction, abandoning with a certain dogmatism what was also its merit. Mathematics today is very creative, but extremely dispersed, and few fields do the necessary work of synthesis and theoretical elaboration. Reflecting systematically on the conceptual and methodological heritage of the twentieth century is all the more indispensable.

The present volume is a translation by the author, with the help of DeepL, of the French edition, *La pensée mathématique contemporaine*, Presses Universitaires de France, 2001. I thank the publisher thereof, Humensis, for allowing me to publish an English version. As a general rule, the translations of citations are ours: for example, the ones of Aristotle's *Metaphysics* are from the French edition by Tricot. However, for the reader's convenience we indicate in the Bibliography, when relevant, reference English translations of the texts. For the rest, the content is essentially unchanged, except for minor modifications aiming at improving and fluidifying the style.

Nice, France  
July 2022

Frédéric Patras

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# Chapter 1

## Introduction



**Abstract** The second half of twentieth century mathematics was profoundly marked by the rise, domination, and decline of mathematical structuralism. It is impossible to understand the current state and the future of mathematics without understanding the corresponding endeavour and going back over its history.

**Keywords** Bourbaki · Grothendieck · Philosophy of mathematics · Structuralism

During the second half of the twentieth century, two major phenomena profoundly marked and influenced mathematics. Firstly, structuralism<sup>1</sup> emerged as the dominant current of thought from the 1950s onwards both from the point of view of epistemology and mathematical practice. It wanted to legislate on the architecture of the corpus and largely inspired the controversial “modern maths” reform. Structuralism systematically favours the architectonics of reasons, global solutions and the greatest degree of generality to particularisms of all kinds, as well as incomplete theories or results with as yet uncertain contours. In other words, it deliberately favours constituted knowledge and systematic research programmes over anything that may be moving and undecided in mathematical knowledge and research. Teaching has long defended the same values and the same concern for order, at the risk of hiding the fact that mathematical thought is also a space of freedom and creativity.

The other striking fact of the last few decades is the extraordinary impoverishment of the philosophical debate around mathematics.<sup>2</sup> There are many

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<sup>1</sup> Structuralism, to which many developments will be devoted in this volume, is understood as the structuralism of mathematicians. It should not be identified strictly speaking with the structuralism of the human sciences [16].

<sup>2</sup> [Transl. note] Since this book was written the situation has evolved: the philosophy of mathematics is changing. However, debates remain most often unsatisfactory as they still take place among philosophers for the most part without genuine contact with mathematicians and current mathematics, making so that, for the most part, there is still no lively philosophical or epistemological debate affecting the way mathematics is done and conceived.

reasons for this: the repercussions of Gödel's theorems, the decline of the great philosophical systems, lack of interest in the classical problems of the theory of knowledge (analytic, synthetic, aprioric character of mathematical knowledge, etc.). Structuralism, for its part, has adopted the idea that philosophical discourse is alien to scientific thought, and has thus contributed decisively to this impoverishment.

Today the failure of the structuralist discourse to be the main driving force behind the development and writing of mathematics has become evident, but we still need to understand the reasons for it and to embark resolutely on new paths. Many attempts have already been made, contributing to an undeniable renewal. Randomness, dynamical systems, chaos, complexity, the mathematisation of biology or economics: these are research themes which, due to the logic of their development, have led to a questioning of the standards of a certain epistemological orthodoxy.<sup>3</sup> This book addresses the same problems, but from a significantly different point of view. It does not include any technical developments, and a reader unfamiliar with mathematics should find his or her way around as easily as a professional mathematician. Rather than reevaluating structuralism in the light of thematic developments and specific mathematical tools (R. Thom's catastrophes, Mandelbrot's fractals...<sup>4</sup>), it seeks to analyse its failure by returning to what is at stake in the debate, namely mathematical thought itself conceived both as a creative activity and as certifying and organising accumulated knowledge. It has often been said that philosophy and mathematics go hand in hand and, in fact, it is impossible to dispense with recourse to the general theory of knowledge in order to understand the functioning of mathematical thought and the laws of its development. The illusion of an autonomy of mathematical discourse is undoubtedly the most serious of the errors committed by structuralism, and it is also the one that we will seek most systematically to denounce.

Moreover, it is impossible to understand the destiny of modern mathematics without going back over its history. The twentieth century was the scene of the greatest quantitative acceleration movement that science has ever known, without the transformations that accompanied it always being understood. Carried away by their impetus, scientists forgot to reorganise their knowledge and subordinate it to exogenous ends. It is often useless to look elsewhere for the reasons of technicist and internalist disciplinary drifts.<sup>5</sup> Other periods were decisive in the constitution of theoretical knowledge, but they were conscious of their epistemological originality and endeavoured to think of their innovative character and its social and philosophical repercussions. Classical and Hellenistic Greece or the Renaissance thus saw science transform itself with a radicality that can hardly be rivalled. The methods of scientific investigation today seem to obey canons that are too well established for their upheavals to occur only at the margins, affecting the volume rather than

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<sup>3</sup> G. Israel describes these conceptual advances in [53].

<sup>4</sup> For an introduction to these theories, see [80] and [70].

<sup>5</sup> That is to say, taking into account only the technical and internal arguments in the scheme of development and organisation of the discipline.

the very idea of science. Even if such and such a fundamental notion should be abandoned, it is reasonable to think that the edifice of science would be little shaken. Thus, for all their importance, Gödel's incompleteness theorems have had very little impact on the day-to-day work of mathematicians.

Does this mean that nothing is at stake? Obviously not: despite the extraordinary evolution of the corpus, little progress has been made on the fundamental questions that humanity must ask of science. The case of mathematics strikes more than any other. The intellectual debate about it seems to have long died out, giving way to the silent work of researchers. The sacred fire which animated Hilbert<sup>6</sup> or Weyl,<sup>7</sup> ready to move or transgress disciplinary boundaries when it was a question of researching the constituent and original elements of knowledge, has gradually given way to an austere professionalism, respectable no doubt, but a little dull and rather annoying when it takes the form of the authoritarian legalism of Dieudonné.<sup>8</sup>

And yet, although mathematicians are largely uninterested in them, there are problems that are waiting to be solved so that the role of mathematics becomes clearer and no longer depends on clichés, whether it be the “honour of the human mind”<sup>9</sup> or, at the other extreme, the idea of an impoverishment of thought and the all too famous slogan: “Science does not think”.<sup>10</sup> What are the meaning and legitimacy of mathematical knowledge? How does the latter fit into our knowledge of the phenomenal world? What is the meaning for humanity of the theoretical

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<sup>6</sup> David Hilbert (1862–1943). German mathematician whose work and scientific aura played a decisive role in the constitution of twentieth century mathematics and, more particularly, of several disciplines such as the theory of invariants or metamathematics. Hilbert is regarded as the founding father of modern axiomatic thought, whose scope and power he first glimpsed. He was also able to highlight the technical and epistemological problems that its systematic use raises.

<sup>7</sup> Hermann Weyl (1885–1955). German mathematician. He fled Nazism and emigrated to the United States in 1933. His works are characterised by the depth of their views as well as their diversity. In mathematics, there are the questions that one asks oneself and those that arise; Weyl was always interested in the latter, leaving aside the “little games of formulas”. A protean genius, he worked on relativity, militated for intuitionism and intervened decisively in the epistemological debates of his time.

<sup>8</sup> Jean Dieudonné (1906–1994). French mathematician, one of the most active members of the Bourbaki group. Author of numerous didactic and synthesis works. An important figure in the history of mathematics, to which he contributed a great deal, acting as spokesman for a certain ‘structuralist’ orthodoxy. Dieudonné’s position on the relationship between mathematics and reality “consists in not having any: according to Bourbaki’s formula, everyone is free to think what she wants about the “nature” of mathematical beings or the “truth” of the theories she uses, provided that her reasoning can be transcribed into the Zermelo-Fraenkel system [the axiomatic theory of sets]” [26].

<sup>9</sup> “...the sole aim of science is the honour of the human mind, and from this point of view a question of numbers is worth as much as a question of the system of the world” (Jacobi [54]).

<sup>10</sup> “Science does not and cannot think; this is even its good fortune, I mean what ensures its own well-defined approach” (Heidegger [41]). If the Heideggerian way of thinking about science is original, though rather disturbing, the use commonly made of it is strewn with clichés. Isolating the expression “Science does not think” from its context (a reflection on the essence of thought) is one of the most detestable.

aspiration that constitutes the highest ambitions of the man of science? All these questions have been asked throughout the century, including when mathematicians believed that they did not have to answer them and favoured the mechanical advancement of knowledge over its insertion in a collective destiny and ambition.

The different ways in which these questions have been dealt with since the birth of modern mathematical formalism at the beginning of the twentieth century are not indifferent to the understanding and evaluation of the structuralist phenomenon, and we shall dwell on them all the more since the possibility of a scientific humanism at the dawn of the third millennium is what really matters here. It alone justifies the need for a renewal of a mathematical philosophy. Everything else is only incidental; the existential meaning of science is what must first be decided. Does mathematics still have something to say or must it be satisfied with its conceptual and methodological autonomy? Should we be content to evaluate it in terms of its methodological performance? Without claiming to provide a definitive answer, we will take note of the ideas and lessons that emerge from a century of history and will try, whenever possible, to identify the paths on which thought could tomorrow embark to reorganise knowledge and its cultural legitimisation. Sciences of facts form a *de facto* humanity, and it is not this humanity that is in the background of any research work, with all that it contains of unsaid hopes.

Some great names will guide us on these paths. Foremost among them is Hermann Weyl. His multi-faceted curiosity, enthusiasm and uncertainties bear witness to the greatest intellectual achievement: that which is not satisfied with ephemeral glory and constantly turns away from what it has just achieved in order to move forward and confront the problems still outstanding. Indifferent to risk, he was able to constantly pursue new scientific objectives when the paths of mechanical and technical development of the knowledge already acquired would have been so easy for him to follow! This Faustian character, this permanent quest for more radical knowledge, may have diverted him from the wise and industrious accomplishment of more immediate tasks with more certain outcomes, but it makes his intellectual personality all the more endearing.

Alongside Hermann Weyl: Alexandre Grothendieck.<sup>11</sup> His works have set a school. Beyond their depth, they strike by their very particular style. Grothendieck's writing is carried by an atypical conception of mathematics, described and theorised in texts whose philosophical scope has remained all too unknown. Their accents often relate them to certain Heideggerian writings. They vigorously testify to the inescapable poetic part that animates scientific work and to this surplus of meaning that formalized writing thinks it is good to evacuate, even though the essence of mathematical thought lies there. Other names punctuate this book:

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<sup>11</sup> Alexandre Grothendieck (1928–2014). Born in Berlin to anarchist parents, he came to France to find refuge at the age of thirteen. His mathematical work revolutionised the disciplines he tackled: topological vector spaces, homological algebra, algebraic geometry... Broadly speaking, the Grothendieckian method of problem solving can be characterised as the search for the optimal conceptual context. Once the adequate conceptual horizon is established, the questions resolve themselves: the method of demonstration should become obvious.

Bourbaki<sup>12</sup> first, a key reference in the horizon of French mathematics and herald par excellence of structuralism. Testimonies analyse the “death” of Bourbaki [15], namely the failure of structuralist thought, in its bourbakist version, to organise the edifice of constituted mathematics in a stable way and, more specifically, the failure of the group, overwhelmed by the abundance of results and concepts of modern mathematics, to write a treatise with an encyclopaedic vocation. We insert these analyses in a historical and critical panorama, which should allow a better understanding of the specificities of Bourbakism. The epistemological choices made by the founders of the group largely explain its relative failure. They had believed it possible to follow a radical axiomatic programme, freeing mathematics from any external constraint. It was precisely on this point that Bourbaki stumbled. While his work is technically irreproachable, the methodological postulates which underlie it, first and foremost the idea that there is an intrinsic and relatively fixed “architecture” of the mathematical corpus, have been challenged by the development of mathematics since the 1950s. Recent practice has thus shown that the edifice of mathematics must, in order to be efficient, regularly revise its internal organisation and shift the attribution of a primitive or founding character from one concept to another according to this evolution. Bourbaki’s position on the relationship between mathematics and reality, which consisted in claiming they “don’t have any”, is another epistemological decision challenged by history: the rediscovery of reality seems to be the hallmark of contemporary mathematical philosophy and one of its most promising avenues of development. Not without reservations, we will conclude by following René Thom in his analysis of the relationship between mathematical creation and the phenomenology of the lived world, or in putting Aristotelian philosophy back on the agenda.

This journey through the history and the topicality of mathematical thinking should make clear that mathematical philosophy can and must continue to be approached with the tools inherited from the philosophical tradition. In other words, the coming era must relearn the procedures for describing scientific creation in terms of world consciousness, intentionality or, more elementarily, spatial or organic intuitions.

I thank Antonella Patras, whose opinions and demanding reading have constantly accompanied the writing of this essay, Jocelyn Benoist, whose ideas played an essential role: the phenomenological orientation implicit in this work is entirely due to him, Dominique Lecourt, to whose advice this book owes a great deal, both in form and in rigour, Pierre Cartier, whose intellectual generosity taught me the optimism of thought. And finally, all those who have given me their support or assistance.

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<sup>12</sup> N. Bourbaki. Group of mathematicians organised as an association under the French law of 1901 governing non-profit organizations. The initial ambition of the group, born in the 30’s, was to base the teaching of mathematics on rigorous foundations and treatises.

## Chapter 2

# Style in Mathematics



**Abstract** The convictions and beliefs related to paradigms and the aesthetic notions associated with mathematical practice can be grouped under the term of mathematical style of thought. The existence of styles is not an accidental phenomenon, but rather the manifestation of general properties of scientific thought.

**Keywords** Kuhn · Paradigm · Style · Writing mathematics

Contemporary mathematics is engaged in a transition movement which has already begun to affect its social status: the role of mathematics as a discipline of school selection is being severely questioned. In parallel, the content of its teaching has been regularly reviewed since the early 1980s in the sense of a weakening of the theoretical components, as if it were now necessary to be wary of the type of formalised abstraction which is nevertheless the hallmark of the discipline. Thus the conquering optimism of the 1950s, which led to the so-called “modern maths” reform,<sup>1</sup> is no longer timely. Mathematical thought no longer claims unconditional universality, as was once the case: the idea of bringing the natural sciences back to exclusively mathematical models according to the canons of classical reductionism<sup>2</sup> has become untenable. As far as the human and social sciences are concerned, they clearly belong to another form of scientificity, but their relationship to mathematics was not always clear; it will suffice to evoke the idea of a mathematical approach to structural anthropology or psychoanalysis.<sup>3</sup>

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<sup>1</sup> Noting the growing gap between the conceptual tools involved in the development of their discipline and necessary both for the diffusion of new knowledge and for any research activity, and the type of mathematics taught in high school and university, mathematicians sought from the 1940s and 1950s onwards to adapt the formal architecture of their discipline (its axiomatic and deductive ordering and, correlatively, the organisation of textbooks and mathematical treatises); teaching was gradually reformed accordingly.

<sup>2</sup> See G. Israel [53].

<sup>3</sup> See C. Levi-Strauss [67] and J. Lacan [61]. In Lacan’s *Écrits*, one will read in particular the “Seminar on the stolen letter” and “Subversion of the subject and dialectic of desire in the Freudian unconscious”.

The consequences of this change in the meaning of mathematics will be all the more uncertain as it will be passively undergone by the scientific community. Historically, the phenomenon is not entirely new. As early as the 1930s, the philosophies of existence questioned the crisis of science and its correlates. The growing influence of technology on the cultural, material and political future of humanity led them to doubt the intrinsic value of a scientific thought which proved to be too indifferent to horror, when it did not put itself at its service. The problems we will be looking at will be much less dramatic, although the same type of issues will be at stake: evaluating contemporary science, and more specifically mathematics, in terms of their collective significance. Proliferation of information, loss of control of global issues by individual and sometimes even institutional players, and finally a feeling of powerlessness on the part of those “excluded from the system”: these are all features of modernity that continue to make sense, *mutatis mutandis*, in the research community.

The parallel has its limits, of course, but the facts are indisputable; like the “global economy”, mathematics is experiencing a dizzying acceleration in its production, which is both a symptom of vitality and fraught with threats. Excessive specialisation, a condition perceived as necessary for academic success, is becoming the common lot of researchers. While online database provide effortless and almost immediate access to almost all the most recent scientific publications, it is not uncommon for a mathematician in a university research department to be unaware of the work and scientific interests of his or her office neighbour. The globalisation of information and research too often goes hand in hand with an atomisation of individual thoughts and cultures, and there is no reason to hope that this movement will be mitigated by a unification of methods and concepts which would eventually make it possible to transcend disciplinary compartments.

Is the future of mathematics likely to be organised into a coherent and global knowledge, as it was still legitimate to hope in the 1970's? Should we rather give up all hope of totalisation in favour of a kind of methodological anarchism? Is it necessary to conceive scientific knowledge as a living organism carrying within it the logic of its evolution and to renounce the idea of transcendental, philosophical or logical norms? Or again, even if it means making a socio-political analogy: should we seek to legislate on mathematical discourse or abandon ourselves to a more “liberal” point of view and bet on the sometimes chaotic processes of self-regulation of knowledge? Each of these attitudes is not devoid of legitimacy and leads to distinct options in the concrete organisation of scientific progress. Thus, while the second may claim the necessary freedom and autonomy of researchers, the first will put forward the mathematical relevance of the notion of research programme.

It is in fact extremely difficult to arbitrate between global and fundamental tendencies, and individual and spontaneous creative processes in the development of mathematics. The idea of a legislation of the mathematical corpus through the fundamental notion of structure,<sup>4</sup> often discussed through its re-elaboration by the

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<sup>4</sup> See “The structuralist current”, Chap. 6 of this book.

social sciences, has governed for a long time a whole part of the mathematical community, among others because of the aura of N. Bourbaki. The relative unanimity that surrounded this idea and the whole Bourbakist endeavour at the time of its apogee, from the 1950s to the 1970s, gradually eroded as Bourbaki's structuralism lost its vitality. The time seems to have come for a critical and systematic reflection on its fertility and its limits.

Addressing these questions, and others that are just as fundamental to the future of epistemology and the social and cultural legitimacy of mathematical thought, will only be possible, however, if it is first clarified what it means in practice to work in mathematics today and what the aims and meaning of contemporary research are: science in motion must have the last word. However, daily experience shows that the general public, even the educated, is virtually unaware of the realities of research and recent developments in mathematical thinking. A few key facts, such as the demonstration of Fermat's theorem, should not conceal the fact that mathematicians are confronted, day after day, with the most total ignorance of the meaning of their discipline and the way it works. Therefore, before anything else, it is necessary to return to the sources and modalities of mathematical activity, beyond the agreed discourses.

According to a standard view, mathematicians are idealists; their activity is based on frequenting and believing in the existence of ideal objects: the line, the circle, real or imaginary numbers. . . . Despite its extreme naivety, such a description is not without historical and philosophical legitimacy. From the first moments of its theoretical elaboration, in the Greece of Pythagoras and Eudox,<sup>5</sup> mathematics was defined as a science of ideas and concepts, an eidetic<sup>6</sup> science which Aristotelian philosophy taught to distinguish methodically from the natural sciences. These patterns of classical Greek and Hellenistic thought continue to organise our perception of mathematics. However, their classical philosophical foundations have been largely lost sight of, so that mathematics and more generally science as a whole, have sometimes had the appearance of titanic edifices without foundations.

The most radical criticisms of scientific modernity have arisen precisely from the questioning of the legitimacy of discourses that have lost their metaphysical foundations. If the type of rationality that is deployed in scientific work is not underpinned by a rationality of a higher order, by a philosophical and ethical conception of reason, nothing can guarantee its intrinsic value.<sup>7</sup> It is not by chance

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<sup>5</sup> Pythagoras (approx. 569–475 BC) is often presented as the first pure mathematician in history. Eudox (408–355 BC) is credited with the theory of proportions (and the theory of irrationality, which was taken up and improved at the end of the nineteenth century to provide a rigorous foundation for the theory of real numbers) and the method of exhaustion (the first stone of the theory of integration).

<sup>6</sup> In the following we will use the adjective *eidetic* to qualify any theory that gives ideas a dimension of reality and objectivity. Eidetic will thus always refer to the (Greek, metaphysical) notion of *eidos* (form, essence. . .). This use ("pertaining to *eidos*") should not be confused with the usual one—the precise recall of something previously perceived.

<sup>7</sup> Value is to be understood here in an ethical sense, but the question of the validity of a science without metaphysical foundations is just as problematic. See E. Husserl [51].



that Heidegger, the thinker of the end of metaphysics,<sup>8</sup> was also the most critical of philosophers with regard to the meaning of science. Heidegger is anything but rationalist, in the sense that, for example, Husserl can be. As such, his thinking seems to us unsatisfactory, more perhaps in the name of a humanism of the heart than because of purely philosophical arguments. Nevertheless, the Heideggerian critique is all the more interesting in that it suggests original paths while at the same time denouncing the defects of modern scientific thought and their existential implications.<sup>9</sup> It thus makes it possible to give content to the adjective “technical”, which all scientists readily use when they want to qualify a demonstration or an idea which is more a matter of dexterity and methodical work than of a truly creative and original activity. Of course, the Heideggerian meaning of the word cannot be reduced to the one scientists give it,<sup>10</sup> but this very distance is rich in meaning and is based on historical and philosophical foundations (the evolution of the concept of technique from its Greek roots onwards). It is this type of fundamental questions, metaphysical in a very general sense, that we believe contemporary epistemologists must relearn to deal with.

Renewing the debate is at this price. If mathematics has an outstanding status in the general scheme of sciences, it owes it, beyond obvious reasons of efficiency, to classical philosophy and to the Greek conception of rationality, and it is to the latter that we must first return.<sup>11</sup> According to it, the primary sciences in terms of quality are those whose principles are the most abstract, those which proceed from the highest degree of universality: philosophy and mathematics. The latter have always served as a touchstone for the general theories of science or knowledge. For a long time they have been ennobled by this function which allowed them to go beyond their own ends (to demonstrate certain statements concerning arithmetic, geometric entities. . . ) by giving the example of an ideal mode of scientificity (perfect certainty, analyticity of reasoning. . . ).

However, modernity has called this model into question. It asked too much of science—more in any case than it could give. The uncertainties of contemporary mathematical philosophy have much to do with this. If the legitimacy of its theories must be reconsidered, we must also violently criticise those who, on the pretext of these evolutions, claim that mathematics has no right to be one of the tutelary disciplines of organised thought and, a fortiori, at the heart of any education worthy

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<sup>8</sup> See M. Heidegger [42].

<sup>9</sup> See “To the Meeting of Reality”, Chap. 9 of this book.

<sup>10</sup> On Heidegger, rationality, technique and scientific modernity, see D. Janicaud [55].

<sup>11</sup> The meaning of the terms “reason” and “metaphysics” has varied over the ages. We understand them here from a very classical point of view: the Greek *logos*, metaphysics in the Aristotelian sense. Thus, the presence of mathematics in Book M of Aristotle’s *Metaphysics* may surprise a modern reader. If he knows the history of physics and the classical interpretations of the Galilean revolution as the victory of science over the medieval Aristotelian vulgate, he will probably think that this presence is typical of a flawed epistemology, which does not know how to distinguish between science (good) and metaphysics (bad). Such a reaction is absurd in view of the Aristotelian text, which must be reread without such prejudices.

of the name. Rationality has certainly been badly abused and, with it, the humanist conception of scientific culture<sup>12</sup> has been questioned, but to renounce making them the necessary foundation of any theory of knowledge and of any educational project would mean renouncing what is most precious in Western thought, the Greek heritage and the optimism of the Enlightenment. It is in this respect that science in general (and mathematics in particular) must be defended against all attempts to devalue it, starting with the most dangerous of all: those coming from scientists who are indifferent to the cultural and educational dimension or the philosophical roots of their activity.

Debates on teaching are particularly important in that they best reflect dominant conceptions of the role of science, but repeated discussions about the preponderant status of mathematics in selection processes have this shortcoming of going to the accessory, to the symptomatic. The limits and failures of the teaching of a discipline are often those of an outdated approach to content. Because, as this book would like to show, mathematics is evolving. Not only in its results, but also in the perception it has of itself, a perception which is today distinct from the one which may have led to the so-called “modern maths” reform. The two most unfortunate consequences of this reform, a formalising drift and the exclusion of the intuitive contents of the manipulated concepts are largely condemned by recent evolutions. However, the ways in which to carry out a reform of the contents and modes of teaching remain uncertain, and the education system seems to be at a crossroads. It knows that it has to take into account the legitimate rejection of systematic abstraction, which is sometimes a little vain, and, what is more serious, which remains most often unjustified in the eyes of pupils to whom the architecture of the administered knowledge remains unknown unless they pursue their studies very far. But it is to be feared that one could go too far in this direction, to the point of losing the sense of fundamental methods such as demonstrative rigour or conceptualization.

We will not dwell further on mathematical pedagogy and its objectives, partly because of uncertainty. It should be clear thereafter that mathematics is resolutely a science of concepts, and it is as such that it should be included in an educational project. For all that, the revolution in computer software is advanced enough to call into question the legitimacy of a whole series of calculatory-oriented teaching methods, which would undoubtedly need to be radically rethought. Perhaps one day it will be necessary to accept the idea that all processes of an algorithmic nature can be left to machines, mathematical education being oriented primarily towards the formation of reasoning. In any case, the situation of mathematics as to its pedagogical and epistemological future is as heavy with uncertainties as it is rich in possibilities, and it is these that we must be interested in.

The first myth that must be overcome in order to move forward is that of uniformity. Mathematics is, from a logical point of view, a profoundly coherent science. The rules laid down in the Aristotelian *Organon*, which describe the axiomatic-deductive structure of reasoning, have always organised mathematical

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<sup>12</sup> That of d’Alembert et Diderot’s *Encyclopédie* [24].

practice, and will continue to suffice as a basis for the essential results. Technical debates on logic and foundations only ever intervene at the margin and, in the end, hardly disturb the overall movement. The great historical polemics, such as those caused by the treatment of the infinite at the beginning of differential calculus or the paradoxes of set theory have made us doubt for a moment the validity of this or that deductive process. However, by forcing us to reflect on certain borderline concepts, rather than causing the whole mathematical edifice to vacillate, they have rather contributed to sanitizing and strengthening it. But unity and coherence do not mean uniformity. Beyond the modes of deduction, there are concepts and intuitions, marking with their imprint the very nature of reasoning. There is a qualitative difference between a treatise on geometry and a text of infinitesimal analysis which will be marked, beyond the difference between the objects of study, by different modes of eidetic aiming.<sup>13</sup> Thus a mathematical demonstration is structured as much by its unspoken words as by its deductive procedures of a logical nature. With very few exceptions, a mathematician by profession, even in the course of a formalized proof, will take for granted a certain number of procedures which, for him, are obvious but would require, in order to be explained in a systematic and rigorous way, an abundance of details and technical ingredients. For this reason, there are several types of mathematical intuition. Intuition is the faculty of orienting oneself in a conceptual field and, if there is a general mathematical intuition, one must also speak of geometrical, analytical, combinatorial, probabilistic intuition... Thus, if a breeding ground of geometrical evidences supports the shortcuts of a demonstration on plane curves, it may remain obscure to the best analyst if his geometrical intuition has not been adequately educated! There are therefore modes of mathematical thinking, which are first reflected in distinct objects of study.

Beyond these thematic distinctions, there are also historical differences that are transversal to the former. Describing them without going into precise textual and technical studies proves to be rather difficult insofar as it is a question of accounting for differences of spirit or conceptual orientation, the apprehension of which remains rather subjective due to the lack of adequate instruments of analysis. It would undoubtedly be convenient to evoke differences in schools: the German algebraic school of the 1920/1930s (Hilbert, E. Noether<sup>14</sup>), the French school of

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<sup>13</sup> A mathematician's perception of objects and concepts such as a circle or an infinitesimal surface element has itself a complex and hierarchical structure. A text in which this structure is very finely analysed (in a fairly general context, not exclusively mathematical) is Husserl's *Psychological Studies for Elementary Logic of 1894* [48]. Schematically, the mathematician's consciousness performs an aiming movement; it is tended towards what it is or becomes conscious of, in our example, the circle or surface element. This aiming is eidetic as soon as what is thus aimed at is treated as an object. When we speak of different modes of aiming, we mean that the relationship of consciousness to its object is structured differently when the object is a circle (a fairly elementary object, which can be represented on a drawing) and a surface element, which is a much more complex concept since it involves the notions of infinity, of limit, of measurement.

<sup>14</sup> Emmy Noether (1882–1935). A German mathematician, she developed the abstract theory of ideals. She played a decisive role in the constitution of modern algebra.

the 1950s to the 70s, the Soviet schools (Moscow, Leningrad), but there is a lack of studies to which to refer in order to support these distinctions. The absence of a history and philosophy of contemporary mathematics which would be able to provide these descriptions and justify them, is sorely lacking here.

But, while the manner is important, the first business of mathematical thinking is the development of its contents. Here too, one must guard against false evidence. Mathematical progress is not purely cumulative and the implicit epistemologies which govern this or that community of mathematicians have a decisive impact both on the way they work and on the content of their productions. This sensitivity of mathematical advances to exogenous parameters of an epistemological nature is decisive, although it is rather difficult to pinpoint apart from a few exceptional cases (let us mention at random, among the great names of “philosophical” mathematics, Leibniz, Bolzano<sup>15</sup> or Cantor). Kuhn’s theory of scientific revolutions<sup>16</sup> gives a first idea of the way in which mathematics progresses in a non-exclusively cumulative way and how it is periodically led to reconsider all its methods and problems. All this will only be worth as a first indication, because if the Kuhnian theory has an obvious relevance for the natural sciences, its application to the pure sciences of concepts (logic, mathematics, syntax, grammar...) is much more problematic apart from certain rather marginal situations. The same remark would apply to all epistemologies developed in the context of confrontations with physics, biology or the human sciences, and whose transposition to mathematics has always proved to be rather unfortunate.

The central notion in Kuhn’s work, which we will allow ourselves to deal with here quite freely and with a concern for fidelity to the spirit rather than the letter, is that of paradigm. It essentially denotes the field on which a scientific theory is deployed, namely on the one hand the theory itself, that is to say the totality of its conceptual achievements, the corresponding technical and written corpus, as well as all the problems that it is legitimate and classical to envisage in order to seek to extend the understanding already acquired, and on the other hand, in a more subtle way, the set of more or less implicit principles and traditions which organise this theory. The essential idea is that a scientific theory always unfolds in a pre-given horizon. It is impossible to go beyond the limits imposed by this horizon without upsetting its organizing principles—without a revolution, a change of paradigm.

The most popular examples of scientific revolutions in the sense of a radical paradigm shift come from physics. Einstein’s restricted and then general relativity could only be developed by renouncing certain regulatory concepts of Newtonian physics. Einstein had to question the very nature of the concepts of space and time to make the mathematical modelling tools he needed simply acceptable from a

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<sup>15</sup> Bernhard Bolzano (1781–1848). Czech philosopher, mathematician and theologian. According to his own judgement, Bolzano appreciated above all in mathematics “the purely speculative parts, in other words that part of mathematics which is at the same time philosophy”.

<sup>16</sup> T. Kuhn (1922–1996). Philosopher and science historian. The *Structure of Scientific Revolutions* [60], his main work, theorises the non-linearity of scientific progress.

physical point of view. Or, more precisely, the realisation that new mathematical structures incompatible with Newtonian patterns of space-time were necessary led to the development of original mathematical methods and models and simultaneously brought about a new conceptual field, within which these structures could make sense.

Within modern mathematics, the relevance of the classical notion of paradigm is less obvious. First of all, it has a character of conceptual autonomy that is not known to other forms of knowledge, which are rooted in natural phenomena and therefore depend much more on our global conception of the world. If we disregard general logic, which is a pure, universal and... empty of content science of the rules of understanding,<sup>17</sup> mathematics (in which we will include here mathematical logic, the metatheory of logic, etc.) seems to find in itself its own foundation. The concept of paradigm can only take on here a restricted meaning and can, at first sight, only coincide with the set of definitions and results of this or that discipline.

Kuhn's ideas therefore seem to be valid in mathematics especially to account for ambiguous epistemological situations, prior to the systematic development of the modern axiomatic method and mathematical logic, the work of Frege,<sup>18</sup> Hilbert or Russell<sup>19</sup> having rendered obsolete the very type of mathematics for which concepts associated with that of paradigm can be of real interest. The study of the contemporary period therefore requires other tools to understand the dependence of mathematical thought on external factors. More precisely, an idea of Kuhnian inspiration such as that of "mathematical revolution" only makes sense in certain situations which are completely atypical from the point of view of the current conception of mathematical progress.

The most fruitful example in this field is a classic one: Euclidean geometry and its radical change of status at the time of the discovery of hyperbolic, spherical or, more generally, Riemannian geometries.<sup>20</sup> When, around 1816, Gauss<sup>21</sup> saw for the first time the possibility of developing a coherent geometry in which the postulate of parallels, which governs Euclidean geometry, would not be verified,<sup>22</sup> geometry had not yet freed itself from its anchorage in reality. In a certain way, everyone then conceived geometry as the science of the pure forms of living space, an idea to which

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<sup>17</sup> General logic is the science of the form of thought in general. It says nothing about the matter of thought: in this it can be said to be devoid of content. For more details on this conception of logic, see for example Kant's *Lectures on Logic* [57].

<sup>18</sup> Gottlob Frege (1848–1925). Mathematician and philosopher. One of the founders of modern symbolic logic.

<sup>19</sup> Bertrand Russell (1872–1970). Logician and philosopher. One of the main architects of the modern theory of the foundations of mathematics, contemporary formal logic and analytical philosophy.

<sup>20</sup> See C. Houzel [47].

<sup>21</sup> Carl Friedrich Gauss (1777–1855). The *princeps mathematicorum*. Extraordinarily creative mathematician. It seems that Gauss was the first to conceive precisely the existence of non-Euclidean geometries. He did not publish his results, true to his motto: *pauca sed matura*.

<sup>22</sup> This is the definition of a non-Euclidean geometry.

Kant gave a systematic turn in his *Critique of Pure Reason* [56], a fact which is not without extremely important consequences. In particular, in Kantian philosophy, geometric and, more generally, mathematical judgements are conceived as synthetic a priori. Therefore, the diffusion of the works of N.I. Lobachevski<sup>23</sup> and J. Bolyai,<sup>24</sup> with which non-Euclidean geometries were really born, was going to provoke a real scientific revolution in Kuhn's sense. It was then over with Kantian epistemology of mathematics, which until then had been part of the "Euclidean paradigm" for which the adequacy between geometric space and lived space was more or less self-evident. Hence a global and systematic questioning of the very principles of geometry, which culminated in Hilbert's work and a completely analytical-formal reformulation of its foundations.

The transformations that modern geometry is likely to undergo are of a radically different nature. They take place within the already organised science itself and, however innovative they may be, they proceed from a conceptual enrichment which does not call into question the constituted knowledge. This does not mean, however, that there are no revolutions: on the occasion of a conference in Bonn in 1992, F. Hirzebruch, a great name in twentieth-century algebraic geometry, was having fun, a bit provocatively, talking about "the French revolution" in connection with a discovery by J-P. Serre in the 1950s.<sup>25</sup> The play on words had not failed to delight the audience and suggested in passing that mathematicians have their own perception of history. The revolutionary character of Serre's work is obviously not part of Kuhnian theory. It does not question any paradigm and is in no way opposed to the cumulative nature of scientific progress. It is this type of mathematical revolution that we will nevertheless have to account for in order to grasp the advances made recently.

Kuhn himself was aware of the limitations and ambiguities of the classical concept of paradigm, as he had introduced it in 1962 in *The Structure of Scientific Revolutions*. In a 1969 postface to his book, he tentatively preferred the less connoted term of disciplinary matrix to the term paradigm or theory: "*disciplinary*, because it refers to the common possession of the practitioners of a particular discipline; *matrix* because it is composed of ordered elements of various sorts, each requiring further specification". He then distinguishes between various types of elements that make up a disciplinary matrix. Two classes of elements are particularly interesting and their systematic study, beyond the few fragmentary indications given by Kuhn, allows us to reflect on contemporary mathematics. The first is the "metaphysical part of paradigms". Kuhn understands by this the fact of collectively

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<sup>23</sup> Nicolas Ivanovich Lobachevski (1793–1856). Russian mathematician, founder (with J. Bolyai, Gauss's work having remained confidential) of non-Euclidean geometry of the hyperbolic type.

<sup>24</sup> János Bolyai (1802–1860). Romanian mathematician who, together with Lobachevski, invented non-Euclidean geometry.

<sup>25</sup> Jean-Pierre Serre (1926,-). French mathematician. He was awarded the Fields Medal in 1954 for his work in topology (the application of Leray's spectral sequences techniques to the theory of fibrations, which allowed the explicit determination of certain topological invariants of spheres: this is the revolution reported by Hirzebruch).

adhering to certain beliefs such as: “heat is the kinetic energy of the constituent parts of the body”. The Cartesian algebraic approach to Greek geometry, and the belief that “all geometry problems can easily be reduced to such terms that it is only necessary, afterwards, to know the length of a few straight lines in order to construct them” [23] can quite probably be considered as a mathematical example of a metaphysical paradigm.<sup>26</sup> Another example, a recent one, is given by the convictions of both supporters and opponents of non-standard analysis.<sup>27</sup>

The other class of elements in a disciplinary matrix are values, of which Kuhn gives several examples. One group of values requires special attention from a mathematical point of view; these are the values that we will refer to as aesthetic. Simplicity, elegance and clarity play an important role in the evaluation of mathematical work. Often it is to them that a great mathematician can be recognised. However, these are notions which are largely the product of individual opinions and matters of taste, even though they may be uniform within a given community or mathematical school.

From now on, we will group together the convictions and beliefs related to metaphysical paradigms and the aesthetic notions associated with mathematical practice under the deliberately vague term of mathematical *style of thought*. An important part of this book will be devoted to the study of mathematical styles through works such as those of Hilbert, Bourbaki or Grothendieck. We therefore do not insist on the scope of the notion, which should be self-evident at the end of these studies. If we admit that the existence of styles is not an accidental phenomenon, but rather the manifestation of general properties of scientific thought, it now remains to understand their origins and philosophical mainsprings.

This is probably easier to do by studying the moments of transition from one style to another. Why, partly independently of technical aspects, do such transitions occur? Can they be explained in a uniform way or is each one a specific phenomenon, linked to particular scientific and historical conditions? Why does a generation of mathematicians spontaneously rally to a new mathematical style, as was the case in France with Bourbakism?

Since the birth and establishment of a new style in a mathematical community is, in addition to its scientific components in the strict sense, a process of collective adhesion, it is reasonable to think that some of its features are not peculiar to mathematics, nor indeed to science in general. In other words, the adoption of a new style by a community is always collegial, and the reasons for this have as much to do with the structure of the mathematician community (charisma of its leaders,

<sup>26</sup> As long as this conviction is not inserted into an apparatus of metageometric demonstrations and proofs that would specify what is to be understood by geometry, reduction, construction: so many terms whose meaning is not self-evident and conveys “metaphysical” components (geometry as a science of space, the idea of reduction as soon as it implies the existence of a “metamathematics” of geometry—not thematized and with very uncertain contours in Descartes—, and so on.).

<sup>27</sup> Some of which can be partly led back to individual convictions about the nature of the infinitely small and the infinitely large. M. Zerner reports on the debates on non-standard analysis in an article (of the same title) in [91].



organisation of scientific and academic decision-making power... ) as with purely rational arguments. In this respect, its understanding must be based on a general theory of transitions between modes of thought (scientific, linguistic, political, etc.), conceived as group phenomena. We found this theory in the first Hegelian writings, those of the time of Bern (1793–1796). That Hegel, the thinker par excellence of the future of the mind, of its progress, of its “phenomenology”, can act as a guide in the understanding of mathematical activity will not come as a great surprise. At most one will be able to be astonished that his first writings are privileged here, all the more so as they deal with the movements of religious feeling. According to the incisive account given by Lukács,

For the young Hegel, the positive Christian religion is a support for despotism and oppression, while the religions of antiquity, which were not positive, were the religions of freedom and human dignity. Their revival is, according to the young Hegel’s ideas, the revolutionary goal to whose fulfilment the humanity of his time is called [68].

If these politico-religious ideas are obviously of little significance from a mathematical point of view, the concept of positivity, which is at the heart of Hegelian argumentation, allows us to understand how generations of mathematicians have been able to perceive the emergence of a new mathematical style as that of a true space of freedom and creation. The militant enthusiasm which surrounded the birth of Cartesian geometry, the Bourbakist epic or, more recently, the development of non-standard analysis, in its political dimension (respectively, turning the page on a bygone era of geometry, mathematics or classical infinitesimal analysis) cannot be explained otherwise.

The idea of positivity is to be understood, as in a legal framework, in its opposition to that of naturalness: contrary to a hypothetical natural religion, a positive religion is imposed by a theological or ecclesiastical framework which claims to substitute itself for reason in the determination of ethical and theological truths. However, Hegel is quick to qualify the scope of such a distinction. The idea of nature is by itself too empty to serve as a touchstone for any judgment on the value of a religion and, among other things, its authoritarian dimension. Rather, what measures the latter is the relationship of a religious discourse to a particular period in the development of humanity. Such a religion may be progressive under given historical conditions, and under other conditions an instrument of oppression. Hence a variant of the idea of positivity: since, in its opposition to the more or less empty idea of naturalness, positivity is a hollow concept (any concrete religion is, in essence, positive, the case of the ancient Greek and Roman religions being quite particular), positivity must measure the adequacy of a religion to the spirit and political conditions of its time. In this respect, the idea continues to apply to all forms of collective thought. As soon as a constituted discourse no longer expresses the needs of its time, is no longer an instrument of progress, as soon as it becomes positive, it must be replaced by a new one, and this is true for religion (cf. the councils) as well as for politics (cf. May 68, Gorbachev’s reforms) or science, from



a technical as well as a philosophical and programmatic point of view (cf. Galileo, Bourbaki<sup>28</sup>).

It is also important to remember from the ideas of the young Hegel that any form of thought initially constituted in a context in which it proposes new instruments of analysis ends up crystallising into a positive discourse. Time alters its adequacy to reality. The truth that unfolded in the original thought corresponded to lived meanings. One only has to look at old geometric manuscripts to see to what extent the way of thinking implemented in them has become foreign to us. And it is not only a question of technique. The fundamental tools of study are preserved during mathematical progress, but they change name and appearance: in the transmission of scientific knowledge we can observe the same phenomena of sedimentation of meaning as in the history of religions. The forward movement of science leads to changes in disciplinary matrices, to use Kuhnian terminology, but these changes, although necessary, are not without drawbacks.<sup>29</sup> Such a geometric theory is first developed in the context of a regular frequentation of a certain type of objects, for example the spheres, ellipsoids and other similar objects (the quadrics) of Euclidean geometry. Gradually, it is enriched by increasingly complex and finer results, which require the introduction of new arguments and technical definitions that are far removed in their very essence from the intuitions that underpinned the theory in its beginnings. These new instruments of study are put in systematic order in new theories, richer without doubt, but which end up forgetting their genesis from a system of spatial evidences. In the case of the geometry of quadrics, it will be incorporated into the theory of quadratic forms, projective geometry and its duality phenomena, the Kleinian theory of transformation groups:<sup>30</sup> so many mathematical domains which can be the object of axiomatic formalizations where the original geometric aspects are lost sight of or considered at the margin. It is not surprising, then, that students who approach the study of quadrics through these sophisticated theories have the impression that they are confronted with a corpus of formulae and definitions whose spirit escapes them. They appear to them as themes imposed arbitrarily, in the name of an architectural cohesion which surpasses them.

To make the part of modernism in the style of exposition and still be able to return to the system of original intuitions underlying a theory is undoubtedly one

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<sup>28</sup> It should be remembered that Bourbaki was not a creator, in the sense that, for the most part, he did not obtain new results. His role was to modernise mathematical discourse and writing and to reorganise the corpus around flagship ideas that were fairly classical (such as the idea of founding the mathematical edifice on set theory), but to which he often knew how to give a new tone (as was the case with the idea of “structures”). In addition, he breathed a new spirit into the research of his time: he created a style. Many great mathematicians had a role similar to that of Bourbaki, but it is more difficult to distinguish between the influence of their research work and their spiritual aura.

<sup>29</sup> Reasons similar to those behind these drawbacks, namely a gradual oblivion of the original meaning of the edifices of modern science, may lead some Christians to regret primitive Christianity. On this nostalgia for the origins of Christianity, see Hegel [40].

<sup>30</sup> For more details on these questions, consult for example the course on projective geometry by P. Samuel [76].

of the major difficulties facing mathematical pedagogy today, because science is condemned to be sterile if it ceases to be based on a full and living intuition of its contents. It is the awareness of this sterility which governs the reactions of rejection of mathematical teaching as a block of gratuitous abstractions devoid of tangible meanings. Opposition to organised scientific knowledge is for many a naive form of rejection of arguments of authority. But, in this matter, a revolt is better than resigned indifference, the exercise of freedom of spirit beginning with the rejection of dogmatisms, even learned ones. The very contents of knowledge are then less in question than the irritation due to a tutelage that students can anyway overturn by becoming part of the game of science, that is to say by acquiring mastery of the concepts that govern it.

However, it should never be forgotten that, even if they are accompanied by a loss of sight of origins, scientific progress requires changes in thinking styles. The demand for unconditional fidelity to origins that may arise from the irritation caused by excessive levels of abstraction in mathematics teaching is dangerous. What sense would it make today to teach Euclidean geometry without considering the algebraic aspects? In the history of any cultural, intellectual or moral entity there are moments of transition when simple survival requires a radical transformation of its leading principles. Otherwise these cease to make sense. If they were not transformed they would only survive for a moment, accepted only by habit and presenting the characteristics of a positive way of thinking: being artificially imposed in a context where they have lost all natural meaning and significance. To believe that a return to the past is possible is an illusion. Hegel says it very clearly about religion: those who defend archaic forms of it, unsuited to their times, are ignorant or dishonest. Ignorant, if they defend them in good faith, because it means that they have remained below the level of intellectual and moral development of their contemporaries. Dishonest, if, aware of archaisms, they continue to defend them.

An ideal science would be in perpetual motion, changing its style when its technical progress or the evolution of the scientific spirit made it necessary, while remaining aware of its past history. Here we must understand history in a conceptual sense: not a factual and heroic story, but a spiritual and conceptual one. It is in this tension between claiming a heritage, a culture and the will to move forward that science best fulfils its destiny. Yet, in mathematics as elsewhere, the abandonment of a discourse that has become positive is never painless. We have said that Bourbaki was the leader of an overhaul of mathematical thinking and the promoter of a new style: structuralism. At the very moment when he was tackling this task, which was admirable both historically and conceptually, regardless of the criticisms that may be levelled at him in retrospect, Bourbaki was resolutely opposed to the constitution of a categorical style of thought,<sup>31</sup> of which it was already clear in the 1960s that it could only end up winning out over the structuralist style. And, what's more: the members of Bourbaki could not ignore on an individual basis this movement of categorification of mathematical thought, for reasons related to their personal

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<sup>31</sup> See "Structures and Categories", Chap. 7 of this book.

scientific backgrounds, just as, to conclude on this point in the Hegelian manner, many clergymen cannot ignore, one hopes, certain detestable archaisms of the Church (with regard to marriage, contraception, certain dictatorships...) without daring to denounce them openly.

To conclude this chapter, the study of two historical examples, already mentioned in passing, will give more substance to the idea of positivity. The first one characterizes both the method and the conquering heroism of Cartesian thought. It is the discovery of analytic and algebraic geometry and its immediate interpretation by Descartes as a transvaluation of the founding concepts of geometry. Historians of science have sometimes wanted to highlight the role of other mathematicians in this discovery, first and foremost Fermat. But, in the name of a misunderstood positivism characteristic of a detailed erudition which loses sight of the essential problems, they have neglected the bottom line, namely the epistemological stakes of an adventure which only Descartes knew how to measure. Generally speaking, the History of Mathematics is today confronted with methodological choices. If it continues to want to move forward in an archaeological mode by exhuming and re-editing marginal texts or by claiming an ever more acute concern for technical precision, it will accomplish an interesting work, but it is doubtful that its work will ever have repercussions other than anecdotal on the history of thought or on modern mathematics. The work of academic scholarship only makes sense in the service of a hermeneutical or epistemological project which, in this case, seems to be lacking.

In the case of Cartesian mathematics, the *Discourse on Method* is explicit. The reform of the sciences undertaken by Descartes had its origins in the observation of the inadequacy of academic knowledge to its own requirements of rationality and clarity, a reproach which extends to mathematics, even if its rigour is not called into question.

Then, for the analysis of the ancients and the algebra of the moderns, apart from extending only to very abstract matters, and which do not seem to be of any use, the former is always so compelled to the consideration of figures, that it cannot exercise the understanding without tiring the imagination very much; and the latter is so subjected to certain rules and ciphers, that it has been made a confused and obscure art, which embarrasses the mind, instead of a science which cultivates it. [22, Discourse on Method].

The development of the use of coordinate systems in geometry and the correlative possibility of using algebra to translate old problems such as Pappus' into algebraic equations<sup>32</sup> give Descartes the certainty that can be observed in the *Discourse* as far as the power of his method is concerned. For, it is a fact, analytical geometry has rendered a whole set of techniques and results obsolete, so much so that the same students who deal without batting an eyelid with the complex questions of this geometry would have a hard time to make an elementary construction using a ruler

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<sup>32</sup> According to Jules Vuillemin: "All Descartes' geometry is intended to solve by a new, analytic and no longer synthetic method, as well as to generalise the Pappus problem" [83]. This problem appears in the famous Pappus Collection, composed around the year 320 of our era; it concerns the determination of geometrical loci related to configurations of straight lines.

and a compass! The “Cartesian revolution” changed the mathematical landscape and imposed its views, while transforming the monumental edifice of classical geometry into venerable ruins, into positive knowledge organised around an outdated global logic. From a technical point of view, the key to the change was the reorganisation, based on the concepts of algebra, of the great divisions of Greek geometry and its classification of problems, consecrated by Pappus, into three genres (planars, solids and “grammics”). Descartes immediately understood that this reorganisation was accompanied by a devaluation of the latter geometry.

There is still a feature of the Cartesian reform that needs to be emphasised, if we want to understand the specificity of mathematical thought and all the ambiguity of its revolutions. The thought of Descartes, in its very opposition to the geometry of the ancients, remains nourished by its ideas and its substance. It was able to impose its rules because it was able to consider the same objects in a new and more fruitful light. As a general rule, the most radical upheaval in mathematics is always limited by some form of immutability of the objects under study and reflects above all a change of attitude and point of view on the part of mathematicians. The natural sciences are not in the same case; in them the great mutations are accompanied by a transformation of the very substance of the objects studied as well as of their mode of apprehension.

The second example concerns the notion of category, already mentioned in connection with Bourbaki, which nowadays has a central role in mathematics.<sup>33</sup> The mathematical use of the word category dates back to the 1940s, apparently inspired by Carnap. The concept itself, axiomatized by Eilenberg and MacLane in two famous articles,<sup>34</sup> leads back to an older mathematical notion, which makes all the sense and scope of categorical language: naturalness. Schematically, any construction which preserves the underlying structures is natural. Thus, in an intuitive sense, but one which could be made rigorous, associating a cardinal number with a finite set is a natural operation. On the other hand, choosing a unit of measurement for the lengths or randomly selecting elements is not: this presupposes arbitrary choices, which introduce an obstruction to the naturalness of the operation. The concept of category makes it possible to study the global behaviour of collections of mathematical objects (one will speak for example of category of sets, topological spaces, groups. . . ) and to study their transformations.

The shaping of this idea, however elementary, and the demonstration of a certain number of formal results on the behaviour of categories provoked a real revolution by making it possible to understand that a whole series of results, sometimes disparate, were likely to be traced back to the same origin: the structure of the underlying category. In other words, mathematicians realized that in some proofs the nature of objects mattered less than the relations between them, the combinatorics

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<sup>33</sup> At least for the algebraic, arithmetic, geometric and topological disciplines.

<sup>34</sup> S. Eilenberg and S. MacLane, “Natural isomorphisms in group theory”, Proceedings of the National Academy of Sciences, 1942 and “General theory of natural equivalences”, Transactions of the American Mathematical Society, 1945, reproduced in [29].

of relations being the only thing that really mattered. It is possible to see in this an upheaval similar to that which the great Hilbertian discovery had caused in its time, in the field of axiomatics and foundation theory: the results of geometry are logically independent of the meaning that can be intuitively attributed to them. A closed system of axioms specifying the relations between various entities of indeterminate nature is sufficient to reconstruct the whole geometry. Thus, in the Euclidean case, by two distinct points one will be able to lead a single straight line. But, even if one decided to call “chairs” the straight lines or “tables” the points so as to deprive the mind of any concrete intuition of the phenomena, the theory would continue to be valid.<sup>35</sup> In the same way, the formal study of categories—regardless of the “nature” of their objects—makes it possible to show that entire collections of a priori radically distinct objects have similar behaviours.

The theory of categories gave rise, like Cartesian geometry in its time, to an original and specific style of mathematical thinking and writing. We have said Bourbaki’s opposition to it, but many have independently diagnosed category theorists as having an overly pronounced and rather useless taste for abstraction for its own sake. As a result, and although the technical importance of categories is now unanimously recognised, their theory paradoxically remains rather relatively marginal from the point of view of the institution and of mathematical teaching, proof, if proof were needed, that the history of stylistic and thematic influences in mathematics is more complex than it seems: the criteria of usefulness and scientific relevance alone are not enough to win the decision!

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<sup>35</sup> See [74, Chapter VIII].

## Chapter 3

# From Plato to Husserl



**Abstract** The facade platonism displayed by the mathematical community is unsatisfactory. Husserl's transcendental idealism seems to be the way today to reconcile mathematical thinking with its intuitive origins.

**Keywords** Descartes · Kant · Husserl · Phenomenology · Platonism

When a transformation is necessary, recurring prejudices often prevent us from seeing clearly what is really at stake. Thinking the new from the old is a difficult undertaking, and a radical break is sometimes easier to impose than a transition without a solution of continuity. Politics is full of these overall movements where the voice of reason struggles to make itself heard in the face of the slogans of the totalitarian ideologies of the moment. It is also easier to make variations on an imposed theme (liberalism, for example) than to question in return the fundamental questions (the essence of freedom), and demagogues and technocrats of all stripes do not care about the essential as far as their discourse seems to take hold of reality. Therefore, it is necessary to constantly reread the great political writers to remember the fundamental principles of public life, and to be able to denounce the semantic abuses committed in their name. The situation is not very different in science, where a number of preconceived ideas can sometimes replace collective thinking and hinder the progress of thought. An error common to most mathematicians, in particular, makes the bed of the worst epistemological errors and legitimizes an absence of philosophical interests on their part. This is the infantile disease of mathematical philosophy: naive idealism or "platonism".

There is hardly a mathematician by profession who does not readily identify himself as a Platonist. There is nothing shocking about that. Mathematicians are in constant contact with ideas. They see them evolve and unfold. They also see them being born and gradually becoming concrete through a perception which, to be intellectual, is no less effective. They are therefore legitimately inclined to speak

of a reality of mathematical objects.<sup>1</sup> But, if we leave it at that and if this only evidence takes the place of a global reflection, it must be said that the mathematician who discovers he is a Platonist does not do much more than Molière's Monsieur Jourdain when he discovers with delight he is speaking in prose. By the way, it goes without saying that it is not Platonism in the strict sense that is in question here, but rather the rough speeches that advocate it, even if they are the work of eminent mathematicians:

R. Thom: Mathematical ideas are produced in our brains to the extent that we think them. But as they exist when we don't think, then they exist somewhere, and not only in our memory: they exist, I would say, also elsewhere...

E. Noël: So they already exist before they have even been discovered?

R. Thom: Certainly! And they are in a certain sense carried out in such and such a case, on such and such a suitable material. This is the old idea of participation that was already with Plato. [81]

The foundations of Platonism need not be questioned: those who claim that mathematical objects are not objects in the full sense of the word testify to a total ignorance of mathematical science, unless they misunderstand the meaning of the word "object". In fact, there is a difficulty here. If we call object a real (actually present) spatio-temporal entity, there is obviously no reason to speak of mathematical objects. However, this concept of object is very restrictive and highly debatable: one would have to decide on the precise nature of spatio-temporal entities and on the meaning of the concepts of presence or reality, problems whose solution is delicate. In the end, consciousness must be attributed a constituent role in the formation of objects and, if one proceeds systematically, there is no reason to insist on a spatio-temporal condition to allow oneself to speak of objects. In any case, this is a conclusion that can be found in authors as emblematic and different as Frege<sup>2</sup> and Husserl.<sup>3</sup> The problem with "Platonism" is therefore not so much its legitimacy as its shortcomings, because it is decidedly having said almost nothing but to call oneself a Platonist, and mathematicians should not believe that they are even for so little: a profession entirely oriented towards the search for knowledge cannot legitimately turn away from the primary questions, all the more so as these questions aim at understanding what nourishes the fascination that mathematics exerts through what exceeds it: elegance, beauty or, in another register, its formidable efficiency.

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<sup>1</sup> Some mathematicians do have an alternative discourse that refers to mathematical "invention". Mathematical objects would not "pre-exist" their "discovery", but would be the result of the mathematician's creative activity. The mathematician would be, in his own way, an artist. There is nothing wrong with this, except that this type of remark, however sympathetic and in no way unfounded, cannot in itself serve as a theory of science. It reveals, in the modalities of its opposition to Platonism, a very rough and naive understanding of philosophical idealism. Its main interest is, in the negative, to highlight the poverty and shortcomings of the philosophical debate around mathematics.

<sup>2</sup> See [31]. The object status of integers is discussed in detail in Chapter VI, as well as the not exclusively spatio-temporal character of objects as a general rule.

<sup>3</sup> See [50]. Read in particular the first chapter of the fourth section on noematic meaning.

What emerged with Plato was the idea of an exemplary status for mathematical objects. Leaving aside the mysticism of numbers or geometric figures in his work, the timeless character of mathematics is affirmed. In various ways, the circle or the theorem of Pythagoras would have an existence in law, an existence prior to any human intervention. Whatever the poetic virtues that can be found in the Platonic mode of philosophizing and a theory such as the one of reminiscence of knowledge, all the ambiguity of the corresponding epistemological contribution is due to an indetermination of concepts that it is very difficult to be satisfied with. If Plato was the brilliant inventor of a philosophical theme: the nature of the objects of thought, the status of the intelligible in its irreducibility to sensitive phenomena, his theory of science is all images, sketches and lends itself to the most varied interpretations. This is its charm and one of its strengths, but it also allows “Platonists” of all stripes to be satisfied with allusive approximations, where modern thought requires coherence and structuring of discourse.

What can be done, then, to go beyond these few vague assertions which form the basis of the mathematical community’s implicit epistemology? For we cannot be satisfied with the atony of the debate surrounding mathematical philosophy, and we must go against the morgue and the quiet assurance of the “mathematicians at work” who neglect to take interest in it.<sup>4</sup> Any scientist who has ever read the *Cartesian Rules for the Direction of the Mind* or Leibniz’s texts on the *Ars inveniendi* or the *Mathesis universalis* can only be nostalgic for a time when the life of thought was not frozen in the conjunction of a soulless technique and systematic contempt for that which transcends it. Such assertions may seem excessive, but the facts are stubborn. Few things go beyond the realm of anecdote or raw facticity in the few recent mathematical autobiographies.

So what can be done? First of all, try to retrace the evolution and drifts of mathematical philosophy. Not of academic mathematical philosophy, but of that which mathematicians live, experience and witness on a daily basis. Everything starts, once again, from the status of mathematical objects. Platonism grants them an existence in their own right. Due to their eidetic nature, the concepts of geometry or arithmetic even have a stronger degree of reality than the objects of the lived world, whose existence depends on the always uncertain validity of the testimony of our senses. Mathematical objects would therefore be transcendent. We can apprehend them, bring them to light or study them, but there is no mathematical creation that is possible in the true sense of the word, as long as these objects pre-exist to the understanding. In spite of its unqualified idealism, it is not sure that this theory can be overcome. At most, a certain number of overly metaphysical features can be eliminated from it. Thus the question of the “pre-existence” of mathematical objects to the perception we have of them, which is only troubling to be badly posed. This

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<sup>4</sup> The expression “working mathematician” is quite misleading. It is common enough to appear in the title of a famous work on category theory (MacLane [69]). Dieudonné uses it to justify Bourbaki’s written work (to provide tools for mathematicians “at work, i.e. engaged in research”) [27]. In certain situations, the expression has a discriminatory content towards mathematicians engaged in marginal or decried research directions. The title of MacLane’s work actually alludes to this usage and to the attitude of the mathematical community towards category theory.



type of question has been sufficiently dealt with by the philosophies of the twentieth century that it is useless to dwell on it for long. Let's just say that the idea of such a pre-existence is based on an erroneous conception of temporality. Thought proceeds from a finite temporality (that of man, mortal) and it is from that starting point that the properties of permanence of being in general and of ideal objects in particular must be examined.<sup>5</sup>

However, if the conclusions of "Platonism" are not to be rejected, it is impossible to admit them without discussion. The difficulty concerns the transcendent character of mathematical objects, modern thought teaching that one must be wary of transcendence: to assert that an object is transcendent is to evade the problem of its constitution. The example of geometry speaks for itself. The naive attitude consists in admitting as self-evident the existence and the ideality of geometric objects, which however is far from being obvious, as we can already see by trying to describe without presupposing them already known the most elementary objects (the point, the line...). The search for solutions to the epistemological problems which then arise must very quickly call upon technical and delicate concepts: pure forms of sensibility, conditions of possibility of an a priori synthesis (as regards the relations of geometry to space and intuition), axiomatisation of geometry, models of a system of axioms (from the point of view of logic), etc. From this point of view, the paradoxes of set theory have marked a turning point in mathematical thinking. They have shown that a definition is not always sufficient to guarantee the existence of a corresponding concept (thus, of the set of sets), and have for long oriented the interest of mathematicians towards questions of logical technique when addressing problems of foundations. Incidentally, "metaphysical" problems (transcendence of the mathematical object...) have been relegated to the background. However, this technical drift was not self-evident, especially since the mathematical specificity of the debates is less obvious than it seems: problems of a very similar nature had already led classical logic and metaphysics to reform themselves at the end of the eighteenth century and to reconsider their relationship to transcendence.

Let us follow again, as in the previous chapter, the paths of theology in order to clarify these ideas and better understand what is at stake for living mathematical thought. The structure of some of the proofs of God's existence is not without recalling arguments familiar to the mathematician, who knows that, in a demonstration, logical inference is only possible under certain conditions. Those which proceed from intimate experience, like the Kantian argument from morality, obviously have a special status as soon as they lead back to irrational arguments. More interesting for us are the proofs like the *a contingentia mundi* one, which go from the effect (the existence of the world) to a permanent and necessary cause (God), but here again the logical reasons interfere with metaphysical arguments which are difficult to account for without much detail. We will therefore concentrate on the logical marvel that is the Cartesian proof, also called ontological proof.

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<sup>5</sup> This does not go without posing delicate philosophical problems, which manifest themselves in particular in tensions between the idealistic (Husserl) and existentialist (Heidegger) tendencies of phenomenology. We refer to [20].

Let us leave it to Descartes himself to explain, in the fourth part of the *Discourse on the Method*, and then in the fifth of his *Meditations*, the deep link which unites his demonstration to the problem of the existence of mathematical objects. We know the movement of the *Discourse*. Descartes starts from the unconditional certainty of “I think, therefore I am”. He then moves from the systematic doubt and uncertainty of knowing to the idea of perfection, which leads him to that of God.

I wanted to seek, after that, other truths, and having proposed myself the object of the geometers, [...] I went through some of their simplest demonstrations. And having taken care that this great certainty, which everyone attributes to them, is based only on the fact they are conceived as obvious, [...] I also became aware that there was nothing at all in them which assured me of the existence of their object. Because, for example, I could see that, supposing a triangle, its three angles had to be equal to two rights; but I saw nothing which assured me that there would be any triangle in the world. [22, *Discourse on method*]

A proof of the existence of God will be necessary to justify the constitutive character of mathematical evidence, whose validity God will guarantee:

...the very thing that I have sometimes taken for a rule, namely that the things which we conceive very clearly and distinctly are all true is only assured because God is or exists. [Ibid]

The Cartesian theory of science therefore requires a principled basis for the existence of mathematical objects, and finds it in God. Ironically, it is a flawed reasoning (the ontological proof), which infers from concept to existence, which is used by Descartes to re-establish an unconditional legality of mathematical discourse.

Coming back to the idea I had of a perfect being, I found that existence was understood there, in the same way as it is understood in that of a triangle that its three angles are equal to two rights, or in that of a sphere that all its parts are equally distant from its centre, or even more obviously; and that, consequently, it is at least as certain that God, who is this perfect Being, is or exists, as any demonstration of geometry can be. [Ibid]

The incompleteness of Cartesian reasoning is of a logical nature: Descartes defines God as the perfect Being. Existence participates in perfection, and therefore God necessarily has existence as an attribute. Since the Cartesian definition of God already implicitly postulates his existence, the ontological proof would be reduced to a syllogism and would therefore be irrefutable if it did not admit in passing that it is sufficient to have a definition to get access to an object or a concept.

It turns out that such a property of definitions is not self-evident; it has been a proven fact in mathematics since Russell's paradox.<sup>6</sup> The paradoxes of set theory are therefore much less far from the ontological proof than it may seem! The limit set which would contain all the sets (the set of sets) and the limit Being provided with all the perfections are two inaccessible concepts, of which nothing allows to legitimize the validity, the fundamental rules of thought being valid for rational theology in the Cartesian way as well as for mathematics.

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<sup>6</sup> Let  $w$  be the predicate: to be a predicate that cannot be predicated of itself. Can  $w$  be predicated of itself? [75].

It is actually quite amusing that two great French mathematician-philosophers, Pascal and Descartes, both rubbed shoulders with theology through scientific arguments without realizing the logical and technical difficulties encountered. Pascal's wager, which is based on a calculation, at the very least audacious, of what we would call today game theory, does not hesitate to engage in an arithmetic of infinity and grace, in a manner much more irreverent towards faith than the Cartesian ontological proof. In both cases, mathematics and logic advance on a ground where their methods cease to be valid unconditionally. The question is posed to determine their field of action and the reasons which make it possible at such moment to accept a reasoning and at such other to refuse its validity. When Descartes returns in the fifth of the *Meditations* to his demonstration of the existence of God, he is aware of the obstacles to the logical relevance of the ontological argument:

...having become accustomed in all other things to making a distinction between existence and essence, I easily persuade myself that existence can be separated from the essence of God, and that thus God can be conceived as not actually being. [22, *Meditations*]

However, if there is a purely ontological problem here, since the relationship between existence and essence is at stake, a question of a similar nature arises in the field of mathematics. What is the content of reality of mathematical objects and can one conclude from the evidence of a concept to the existence of the corresponding object? Descartes, there again, knows the difficulty but evades it, perhaps because as a good mathematician he cannot really question the eidetic reality of the objects of geometry or arithmetic: if the recourse to God of the *Discourse* does not convince, the systematic doubt hardly holds in front of the mathematical evidence.

As, for example, when I imagine a triangle, even though there is perhaps nowhere in the world outside of my mind such a figure, and there never was one, there is nevertheless a certain nature, or form, or determined essence of this figure, which is immutable and eternal, which I have not invented, and which does not depend in any way on my mind. [Ibid]

At the end of this incursion into Cartesian work, two remarks are in order as provisional conclusions. First of all, it is vain and unfortunate to try to circumscribe the scope of mathematical philosophy. To separate what belongs to metaphysics and to the theory of science presents technical advantages, and avoids scientists many hazardous debates, but sacrifices in the process what is most precious: the deep unity of thought and knowledge. Confrontations of fields of thought, however far apart, can allow progress, even when it is a question of understanding the contents of technical problems (for example, the general scope of Russell's paradox). That Descartes chose to speak *simultaneously* of mathematics and metaphysics in his writings is not gratuitous: the solution of the problems he faced was at this price. It is anything but certain that modernity is right when it is reluctant to explore similar paths.

The second remark refers to the Cartesian method. If one rejects the theological conclusions of the *Discourse* or the *Meditations*, the question is raised of delimiting the nature of mathematical concepts and the conditions of their validity. It is a question of restoring their objectivity, since the latter is still in the scope of Cartesian

systematic doubt but no longer guaranteed by the existence of a God who cannot deceive. The question is not only a matter of mathematics or formal logic, since it is ontological and affects all areas of conceptual thinking. It is a question of establishing a legislation of reason and criteria to distinguish those fields of thought where reasoning can claim to lead to certain conclusions. Kant's critical thinking proposes such a legislation.

[His Critique] is that of the power of reason in general, considered in relation to all the knowledge to which it can rise independently of all experience; consequently the solution of the question of the possibility or impossibility of metaphysics in general, and the determination of its sources, its extent and its limits, all this following principles. [56]

Nineteenth century science has challenged many of the conclusions of Kantian thought.<sup>7</sup> However, the latter continues to provide a framework within which to formulate the fundamental questions that mathematical philosophy will have to face again if it is to avoid taking refuge in the problems of the metatheory of logic or the history of thought. Like any great system, Criticism synthesizes the achievements of an era and traces the great delimitations that will structure later thought. Of its themes, the richest of developments and the most enigmatic from the point of view of mathematical epistemology is undoubtedly transcendental idealism, with which we will be constantly confronted in the rest of this book, either explicitly or indirectly, through the choices made by such or such contemporary mathematical school.

As for the nature of mathematical concepts, until the great movement of arithmetisation of analysis, which marked the end of the nineteenth century, there was a certain indecision as to the choices to be made in order to get a foundation for numbers or geometry. Intuition and obviousness had become questionable, in Kant's time and beyond, and no thinker worthy of the name could simply go back to them without further ado. Mistrust of intuition only increased after Kant, moreover, as its limits became apparent, especially in analysis, where unforeseen and "monstrous" phenomena manifested themselves with regard to naive intuitions of the continuous or of space (think to Peano's curve).<sup>8</sup> The main options, which can be found up to Frege's time in literature, consisted, for arithmetic, in leading back number to space, time or subjectivity. As for geometry, the solution was simpler, since it seemed sufficient to refer to the nature of space—even if the latter remained problematic.

The arithmetisation of analysis, from Weierstrass<sup>9</sup> to Frege, and the Hilbertian axiomatisation of geometry, which are today an integral part of the basic culture of

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<sup>7</sup> According to the *Critique of Pure Reason*, the mathematical way of thinking is synthetic a priori. Mathematics is inseparable from the pure forms of sensibility. These two theses have been defeated, both because of mathematical advances proper (the discovery of non-Euclidean geometries, the study of the foundations of analysis) and because of advances with more resolutely philosophical content (as with Frege's *Foundations of Arithmetic*) [31].

<sup>8</sup> Giuseppe Peano (1858–1932). Its curve fills a square.

<sup>9</sup> Karl Weierstrass (1815–1897). German mathematician, whose name has remained associated with the idea of rigour in analysis.

any scientist, make the stakes of previous work difficult to perceive in retrospect. However, it is perhaps an illusion to believe that one has made much progress on these questions by having been content to give, as has been done, axiomatic bases to mathematical practice. The results obtained by modern science are more certain than those obtained up to the end of the nineteenth century, but this progress in certainty is due first of all to syntactic and symbolic advances. Be that as it may, the question Kant faced cannot be evaded: how can the a priori validity of mathematical results be justified without resorting to the subterfuges of pre-critical metaphysics? The axiomatic-formal answer (referring everything to formal non-contradiction), apart from solving nothing in substance (according to Gödel's theorems), in any case considers only one aspect among others of the question, as non-contradiction or exactness are not the only possible measures of truth!<sup>10</sup>

In Kantian philosophy, pure logic is a science of pure forms, a kind of syntax of reason incapable of harboring the slightest object or content.<sup>11</sup> Here again, the choice made by modernity to attribute to formal logic a founding role is debatable, despite the soothing certainties it provides. Greek thought was already aware, with Aristotle, of the possibility of returning classical geometry to a minimal system of postulates, but, more nuanced, it did not neglect to seek in metaphysics the epistemological complement indispensable to arrive at a satisfactory theory of science. Contrary to modern thought, it wished to account for mathematics in the totality of its manifestations and, in particular, its real and factual content; after all, the success of mathematics is first of all due to its operative character!

Since logic is indifferent to content and meaning, the sources of mathematical thinking must be sought elsewhere. For Kant, pure reason is the faculty that contains the principles by which it is possible to know something entirely a priori. An abusive use of this faculty has led metaphysics to the worst errors: we have seen what happened with the proofs of the existence of God, but all a priori ontological deductions would prove equally questionable. The usefulness of critical philosophy is therefore first of all to assign boundaries; its function is to preserve thought from any error of method and to set limits to speculation. As such, critical philosophy is necessarily transcendental: "I call transcendental all knowledge that does not generally concern objects, but our way of knowing them, as far as this is possible a priori [56]."

As far as geometry is concerned, the very possibility of a transcendental determination of its contents is much more difficult than it seems. If one refuses, like Kant, to assign an ideal and unquestionable existence to geometric objects from the outset (as Platonism does), one must take the measure of the relationship of geometry to the intuition of space. Kant thus seeks to reconcile the intuitive origin of geometric concepts with their aprioric status: the aim is to explain the almost phenomenal existence of circles, triangles or spheres that can be realised in our

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<sup>10</sup> The concept of truth is discussed in sect. 39 of the third of Husserl's *Logical Investigations* [52], echoing Aristotle [3,  $\theta$ , 10].

<sup>11</sup> See footnote 17 in Chap. 2.

intuition of space, without giving up a certain idealism. The Kantian method is organised in a very natural way around the idea that mathematics is not an analytic science, but a synthetic a priori science. It is not analytic, because analysis is syllogistic in essence and never allows thought to go beyond the formal properties of concepts already known; mathematics proceeds instead from a movement of extension of concepts which links the known to new terms that would be irreducible to the old.

However, recognising the synthetic character of mathematical discourse poses more problems than it solves, because synthesis proceeds by extension of the conceptual field and can have no other legitimacy than that of eidetic intuition. Kant seems at this point to have to found geometry on the system of spatial evidences, with all that is metaphysically suspect in such a process. It would mean returning to Descartes. Kant only gets out of the aporia by a real philosophical sleight of hand, as admirable in his craftiness as the Cartesian ontological argument. Since intuition cannot legitimately found geometry in the system of critical thought, the terms of the debate must be shifted. After all, “everything in nature happens according to rules, although we don’t always know them” [57]. Thus, reason is subject to imprescriptible rules, those of general logic. In the same way, geometry is the science of the rules of the intuition of space. Or, to put it another way, our spatial intuition is conditioned a priori by the rules of geometry.

Let’s take the example of the triangle. All mathematical philosophies ask, at one time or another, how it is possible to pass legitimately from a triangle drawn in pencil to the corresponding intuition or concept. How is the abstract triangle related to the drawn triangles? The basis of the Kantian idea, more familiar to our minds than it seems, is that one can only speak of a triangle by indicating a figure drawn on the board if one already has the corresponding concept in mind, in all its determination and logical coherence. In other words, when we think that we define a geometrical concept by proceeding by abstraction from a figure, we are wrong. For this definition to be simply possible, our mind must already have the prescience of a universal object whose figure drawn in pencil or chalk only testifies to its existence. We recognise in this drawn triangle something like the material trace of one of the fundamental structures of our spatial intuition. The possibility of defining an abstract triangle is always immanent to our daily experience, and when we formulate a linguistic definition of the corresponding concept, we only realise this possibility. There would therefore be many rules more general than those of formal logic, which would justify most of the processes that are at the heart of real mathematics and the immanence of the idea to the intuitive object.

These few remarks on Kantism aim, like this whole chapter, to give, without exegetical concern, an overview of the philosophical problem of the constitution of knowledge and mathematical concepts. The popular epistemological literature has often been full of noise and fury, with a ceaseless talk of revolutions, of discoveries that would upset the very nature of mathematical activity or its relationship to reality. Even the vocabulary designating a certain number of constructions or problems with an epistemological dimension bears witness to this: chaos, turbulence, catastrophes! But behind all these phenomena, the great questions of mathematical philosophy

remain and it is distressing for the convinced mathematician, the one who lives in the intimacy and mystery of concepts, who knows their disturbing origins, who experiments every day with the uncertainties of intuitive knowledge, to find that what makes all the flavour and the scope of his science is neglected in favour of accessory questions. Rereading Aristotle, Leibniz or Kant, without concern for pedantry and with a certain naivety, is the way to get back in touch with a true philosophical questioning, made of astonishment and uncertainties, and accompanied by the feeling that the essential is at stake, that we are closely touching the springs of thought. The more technical problems come only afterwards.

Let us therefore continue reading Kant in this spirit of openness to a certain number of questions, which will come up again and again when it comes to understanding how mathematicians such as Bourbaki, Grothendieck or Thom were able to conceive and expose their creative activity. In the *Critique of Pure Reason*, geometry intervenes in particular in the chapter on Transcendental Aesthetics, which is the science of the principles of sensibility. If we abstract from sensibility everything related to its objects, the idea of a pure form, susceptible of a priori knowledge and independent of any concrete testimony of the senses, remains as a limit idea. As such, the originary representation of space is an a priori intuition and not a concept. Moreover, space is a pure and necessary a priori representation.

It follows from this that, in relation to it, an a priori (non-empirical) intuition serves as the basis for all the concepts we form. Thus all geometrical principles, such as this one, for example, that in a triangle the sum of the two sides is greater than the third, do not emerge from the general concepts of line and triangle, but from intuition, and from an a priori intuition, with apodictic certainty. [56]

The Transcendental Exposition of the Concept of Space completes the description of the origins of geometry in its intuitive and yet certain and aprioric origin.

Geometry is a science that synthetically and yet a priori determines the properties of space. What must therefore be the representation of space for such a knowledge to be possible? It must be originally an intuition [...]. But this intuition must be in us a priori, that is to say prior to any perception of an object, and, consequently, be pure and not empirical. Indeed, geometrical propositions such as these for example: space has only three dimensions, are all apodictic, that is to say they imply the awareness of their necessity; but such propositions cannot be empirical or experiential judgements, nor can they derive from it. [56]

Critical philosophy thus accomplishes the feat of reconciling the intuitive character and the apodictic necessity of mathematical results and concepts. A clear progress is made with regard to naive idealism and even with regard to Cartesianism, where the legitimacy of recourse to obviousness remained subordinate to the existence of God, that is to say essentially to a postulate affirming the validity of the intuition. Yet Kantism, it has been said, has survived badly. As far as mathematics is concerned, the Kantian error is easily identifiable but difficult to repair. When geometry is considered from the point of view of Transcendental Aesthetics, everything rests on an unconditional identification of geometric intuition with the intuition of space. The situation is analogous as far as arithmetic is concerned, since “the intellectual concept of quantity and multiplicity is only born with the help of the concept of time” [58]. The whole nineteenth century has been at pains to establish



the falsity in principle of such arguments. Intuition cannot be hypostasized in this way.

Frege, at least in his early days, was undoubtedly the philosopher who best succeeded in remaining faithful to the fundamental ambitions of Kantism while radically questioning his theory of mathematics. This is particularly evident in the *Foundations of Arithmetic* of 1884, where he developed what was to become the standard method for the logical definition of natural numbers. The post-Fregean mathematical logic is much more debatable. It has lost contact with metaphysics, and with it a form of conceptual dignity. Therefore, fragmentary thought, this part of mathematical philosophy which is born with the systematic development of mathematical logic will be of little interest to us. In the galaxy of the theories of science formulated during the twentieth century, we prefer the one that comes in direct line with the transcendental Kantian idealism and seems to us to better respond to the legitimate aspirations of an accomplished mathematical philosophy, concerned both with thinking the demands of form and with the rooting of mathematics in the experience and life of thought. It is the transcendental Husserlian phenomenology which claims the heritage of the idealistic philosophy, but submits it to the requirements of modernity.

In Kantian philosophy, the general transcendental object and the particular transcendental objects are obtained as ideal limits of the corresponding concepts, empirical or not. Thus, the transcendental object in all its generality is obtained by making abstraction in the concept of object of all that is empirical or accessory. The transcendental object is thus “the most universal object”, an idea which is at the same time the most extended and the most devoid of content. It is what remains of objects once they have been stripped of all their particular determinations. It therefore has an existence and an aprioric necessity independent of any concrete experience: its constitution and its study are indeed a matter of transcendental idealism.

Successively remove from your experimental concept of a body all that it contains empirically: colour, hardness or softness, gravity, even impenetrability, there always remains the space that this body used to occupy (now completely vanished) and that you cannot make disappear. In the same way, if from your empirical concept of any object, corporeal or not, you take away all the properties that experience teaches you, you cannot however take away the one that makes you conceive it as a substance or as inherent to a substance. (Kant [56])

The transcendental object is a pure form. The type of intuition which corresponds to it is not an intuition in the usual sense of the term; it is an intuition which does not concern objects in the usual sense but the form, the structure of our relationship to these objects :

[The transcendental object] is precisely what structures the phenomenal field—of what appears to us—inasmuch as it is the field of our knowledge itself. Phenomena are representations which as such relate to objects. The “transcendental object” is the object of the phenomenon, or, in other words, for the phenomenon, the structure of the relationship to the object. (Benoist [8])



Kant only partially exploits the wealth of concepts thus uncovered. In the field of mathematics, favouring the point of view of transcendental aesthetics, he does not conceive the idea of a transcendental constitution of mathematical concepts. It is to Husserl's credit that he was able to formulate a coherent programme for a transcendental idealism that gives science its rightful place. In Husserl's thinking, mathematics once again becomes the touchstone of many transcendental reasonings and guides the progress of eidetic knowledge. Mathematics once again becomes a privileged mode of access to the normative idea of pure reason. Husserl's phenomenology is indeed profoundly rational and seeks to reconcile modernity with an increasingly abstract and distant science.

Phenomenology, in the form given to it by Husserl, did not have the success it could legitimately hope for, perhaps because its concern for uncompromising intelligibility imposed on it a philosophical style that was too technical and off-putting, whereas philosophies of Being or epistemologies at little cost seduced the public with their reductive simplicity or poetic charm. However, where Descartes brought mathematics back to evidence and, beyond that, to a divine guarantee, where Kant returned it to the structure of our intuition of space and time, Husserl points the way of reason. Mathematical objects are transcendental: they have an aprioric necessity, which emerges in the work of the mathematician.

What does this mean? In his work, the latter encounters objects, typical structures. For example, circles or triangles or even the first natural numbers, which can be experienced in everyday life without necessarily being recognised as abstract concepts. In the Husserlian model of the constitution of mathematical ideas, the transition from the phenomenal circle to the idea of the circle takes place according to Kant's model of the constitution of transcendental objects. From our experience of traced circles we proceed to the idea of the circle but there is a solution of continuity from one to the other which is precisely the transcendental moment of knowledge. The obtained idea is pure and has all the aprioric necessity required because, even if it proceeds from experience, it is logically independent of it: once the idea of the circle is constituted in all its generality, it becomes independent of this or that particular intuition. When the mathematician has mathematical concepts at his disposal, he is assured of their existence without having to return incessantly to the intuitions from which he was able to abstract these concepts.

In this form, phenomenology reconciles mathematics with its intuitive origins. The transcendental object needs a prior experience, a ground of empirical intuitions from which the corresponding transcendental objects will be constructed. There is no mathematical creation without the conjunction of an intuition and a work of conceptualization. It also reconciles mathematics with its ideal and necessary character. Finally, it reconciles the logical and intuitive origins of contemporary mathematics by explaining how formal logic is inseparable from "transcendental" logic. A consequent theory of science must account both for the logical foundations of post-Fregean mathematics and for the intervention of intuition in concrete mathematical thought. This presupposes a reconciliation of analytic and synthetic procedures in scientific progress. Transcendental idealism, by its very nature, justifies the synthetic character of mathematical activity, which refers to the way

in which transcendental objects are constituted from experience, while recognising the necessary and aprioric character of mathematical concepts which is expressed, among other things, in their formality.

All this will become clearer through an example. It has already been explained how to form the idea of the transcendental object in all its generality. Let's see how a particular mathematical object is formed in the transcendental mode: the concept of composition law.<sup>12</sup> According to the rules of transcendental idealism, in its phenomenological version, the idea of composition law must be obtained from a concrete phenomenal experience. The first example which comes to mind is that of numbers: to add is to relate two given numbers or quantities to a third, their sum. Other examples, less specifically mathematical, show that the idea of composition law is not internal to mathematics, but is a structuring concept of all human activity (although one is not often aware of manipulating composition laws, when everyday life leads to do so). Let's take the example of colours: suppose I have a certain amount of white paint and an equal amount of red paint. I can, by mixing these colours, obtain pink. As in the case of numbering, the creation of new colours from given pots of paint involves combining various entities to obtain an entity of the same type (in one case a number is obtained from two numbers, in the other a colour from two colours). Hence the idea of composition law: it is the "transcendental object" obtained by disregarding in the two previous examples what is in one case specific to numbers, in the other to colours. A composition law is thus an operation associating to any pair of objects of the same nature a third object of the same nature (a number, if one starts from two numbers, a colour, if one starts from two colours).

What should be emphasised is that there is always a certain flexibility in the process of defining a mathematical concept from a given experience. The movement of transcendental constitution of objects is not univocal. Starting from numbers, it comes quite naturally to mind to consider definitions of the same type as the laws of composition taking into account certain specificities of the addition process. Hence the definition of laws of composition provided with additional properties: unitary, associative, commutative, etc.<sup>13</sup> In a more subtle way, more general concepts than the one of composition law are conceivable, for example by allowing only the elements of certain couples, and not necessarily all of them, to be composed. In any case, these new mathematical objects are, of course, axiomatically definable, but they refer to fundamental intuitions, which intervene in everyday life without their structuring role for thought, both empirical and rational, being consciously perceived. Mathematical ideas inhabit the lived world!

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<sup>12</sup> If  $E$  is a set, a composition law on  $E$  is an application  $f$  from  $E \times E$  to  $E$ , i.e. an application which, to any  $(x, y)$  pair of elements of  $E$  associates a (single) element of  $E$ , denoted  $f(x, y)$ .

<sup>13</sup> See any elementary algebra textbook for more details, e.g. [11, Algebra I].

To conclude this chapter: the facade platonism displayed by a whole section of the mathematical community is decidedly unsatisfactory. Its fundamental choices are valid, and it is probably impossible to think seriously about mathematics outside of a certain idealism. But the contours of this idealism must imperatively be clarified. The transcendental Husserlian idealism seems to be the way to reconcile today mathematical thought with its intuitive origins.<sup>14</sup>

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<sup>14</sup> The last two chapters set up an epistemological and hermeneutical framework which will be used implicitly in the rest of the book. For the sake of simplicity and to avoid entering into necessarily long and delicate descriptions, we will not insist much on the possibility of explicitly interpreting this or that idea of modern mathematics in transcendental terms. It will suffice that this possibility has been made manifest through the example of the laws of composition. A similar remark applies to the idea of style. Rather than talking about Gauss's "style", Grothendieck's "style" and the constituents of the corresponding "disciplinary matrices", insisting each time on the possibility of understanding the word "style" as in the first chapter, we will simply describe the work and thought of Gauss or Grothendieck. Although not always methodically explained, the link between these descriptions and the ideas that have been put forward so far should be clear.

## Chapter 4

# The Origins of Modern Mathematics



**Abstract** The creation of modern mathematics standards for definitions and proofs, as well as for the organisation of the corpus, took a long time. The analysis of preliminary stages leads to highlight the gradual emergence of a new understanding of mathematics in the work of Gauss, Galois, Dedekind and Frege, among others.

**Keywords** Dedekind · Galois · Frege · Gauss · Transcendental idealism

Modern mathematics could be characterised by the universal dominance of the transcendental method. Until the beginning of the twentieth century, it was often lost in a maze of definitions, modelled piecemeal on the objects being studied. The development of a new approach made it possible to organise a situation that had long remained too chaotic. The spirit of the method is to abstract from the objects studied their formal substance, in the way that the process of transcendental abstraction releases the concepts from their empirical roots. The example we gave previously (the notion of law of composition) is elementary, but it perfectly illustrates the method's springs: from diverse situations, sometimes without apparent links, to extract concepts, universal structures, which can allow to deal simultaneously with questions coming from a priori distinct domains. Algebra has been the driving force behind the realisation that it was necessary to put an end to the ad hoc treatment of problems. From each one of them, it is necessary to extract the universal elements each time, and this with the greatest possible degree of generality. This generality is not in vain, as long as the mathematician gains in lucidity and understanding.

The method is specific to mathematics only in some of its features, the most technical ones, because, to choose an example at random from the configuration of the humanities, when political philosophy has drawn out of the jumble of institutions and particular jurisdictions concepts such as those of state, constitution or system of government, which today organise our perception and understanding of public life, it has proceeded by identifying in different political systems invariant and (almost) universal structures. In the same way, concepts such as those of group, topological space, application or continuity organise today the perception and understanding that a mathematician has of his or her discipline, and by mathematical culture, we

must now understand, more than factual knowledge, the mastery of a significant number of such structuring concepts.

Yet the birth of modern mathematics was difficult, and it would be perilous to try to assign it too precise an origin. Bourbaki's theorization of the structural method in the 1930s and beyond only completed a movement that already had its credentials and which only lacked to be understood and described in all its innovative radicalism. Delimiting the sources of its organizing concepts is a dangerous game. The modern method is often found in the works of the great mathematicians of the seventeenth and eighteenth centuries but remained implicit in these works: it would have needed to be made explicit in order to be recognized with certainty. Leibniz's mathematical works and his quest for a *mathesis universalis* are thus teeming with original concepts, which have all the universality and abstraction one could wish for. Leibniz had the genius to understand the springs of the transcendental method: to go directly to the most general notions, obtained by identifying typical and regulating structures in a class of problems. But this part of his mathematical philosophy was too far ahead of the technical and conceptual level of his time to be adopted by his contemporaries and create a school. Thus it took more than two centuries for Leibnizian programatics to revive, albeit in forms that had lost much of their metaphysical component.

The most striking case of a mathematician whose work bears the mark of the structural-transcendental method is undoubtedly that of Gauss. It is difficult not to recognise in it some of the modern tools already identified in all their eidetic content, and deployed with all the power and universality which are characteristic of the method. Gauss's ideas on the nature of mathematical work are close to those which Bourbaki would later defend. One of the characteristic features of Gauss is the plurality of his scientific interests. He was appointed director of the Göttingen observatory in 1807, following the approximation calculations that led to the discovery of the planet Ceres. He held this position as an astronomer for some forty years, making major contributions to optics, the study of magnetism and mechanics. Gauss is one of those few universal geniuses, from Archimedes to Poincaré, capable of embracing in thought the field of scientific activities of an era and intervening in it in a decisive and protean way. It is not surprising that his work testifies to the profound unity of mathematics by establishing parallels and links between seemingly distinct phenomena. An example among others of his ability to consider the same problem from different angles and to resort, for the same demonstration, to several orders of intuition (geometrical, analytical, topological...) is given by his proofs of the fundamental theorem of algebra.<sup>1</sup> From 1799 to 1850, Gauss published a total of four demonstrations, substantially different in nature.

His works on geometry are among those that shook the Kantian-type certainties about the adequacy of our intuition of space to our sensibility. He began very early in his mathematical career to reflect on the nature of space and on the postulates of Euclidean geometry. It seems that he did not question the regulatory idea that

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<sup>1</sup> The decomposition of a polynomial with real or complex coefficients into irreducible factors.

the real world has a well determined geometry, but it is only almost a century later, with general relativity, that this idea was definitively abandoned in favour of a local conception of the structure of the universe. The first, however, he understood the irreducibility of the postulate of parallels and the possibility of non-Euclidean geometries. The study of problems of curvature linked to these researches led him to consider the possibility of an intrinsic geometry of surfaces, a great and beautiful discovery. Concepts such as that of geodesic curvature were later the starting point of Riemann's work<sup>2</sup> on differential geometry, itself at the origin of the mathematical formalism of general relativity.

Another discovery has shaken up disciplinary barriers and a certain narrow conception of mathematical intuition in its relationship to formalism (especially the algebraic one): the geometry of complex numbers. Their graphical representation had already appeared, before Gauss, in Wessel's work, but this work, which had appeared in the acts of the Royal Danish Academy, had remained confidential. Gauss was early aware of the usefulness of this representation, but did not use it until quite late (1831) in his published works. Here again, he did not hesitate to take the step and identify mathematical objects totally distinct from an intuitive point of view, namely points in the plane and numbers obtained in the search for the roots of polynomials with real coefficients and whose existence had long been held suspect. The algebraic structures underlying plane geometry impose this unnatural identification: form prevails over the intuitive content of the representations, although our intuition can be transformed and adapt itself to new ideas and uses. The geometrical representation of complexes is nowadays an integral part of these unquestioned evidences which are primary components of a mathematics student's baggage.

This is one of the other great epistemological intuitions attributed to Gauss: the mathematician must disregard the nature of objects and the meaning of their relationships. The important thing is the form of these relations, that is to say, to adopt a modern terminology which is, as we shall see, not without ambiguity, their structure. Strictly speaking, this idea of a duality between objects and forms or between objects and structures is debatable: the form of the relations between objects is itself an object of a higher order. All these ideas are in germ, in one form or another, in Gauss. In the *Disquisitiones arithmeticae* [33] and in the course of his other arithmetical researches, he introduces composition laws within the framework of a calculation with classes whose methodology is, up to the terminology, very modern. These calculations with classes, whether congruence classes of integers modulo an integer or classes of quadratic forms, put in place without saying it the foundations of the theory of finite commutative groups. There is here, as at many moments in mathematical history, a difficulty; there is a great risk of attributing to Gauss results in a form which he did not conceive in all its generality. There is indeed in his work the essence of theory, but how to evaluate what lacks: a suitable

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<sup>2</sup> Bernhard Riemann (1826–1866). His work had a decisive role in several fields: topology and differential geometry (Riemann surfaces and spaces), integration theory (Riemann integral), zeta function.

language developed to account for the exact nature of the objects considered? In the transcendental perspective, the precise identification of concepts plays a decisive role, as it delimits the ground on which various theories can be unified. It alone makes it possible to identify, from the specific elements of the problems considered in a given mathematical situation, the universal that can be used in other situations.

Another trait of Gauss's character is his concern for demonstrative rigour. This preoccupation would run through the whole of the nineteenth century, before canons of writing and an adequate formalism were agreed upon. Gauss anticipates in particular the efforts of Cauchy<sup>3</sup> or Abel<sup>4</sup> in this matter, and is wary of resorting to intuition when it is not supported by precise rules. This theme of rigour underwent an astonishing evolution during the twentieth century. Despite all the insistence on the need for mathematics purified of any recourse to intuition and for flawless demonstrations, it is now clear that mathematics would have made much less progress in recent decades if mathematical physics, at the origin of many of the fundamental results obtained recently in pure mathematics,<sup>5</sup> had not often continued to work as the Bernoulli did, going to the essential questions while neglecting to base its results on formalism and indisputable proofs. In other words, the normative idea of rigour would not be an unquestionable ideal which should be valid in all circumstances. It could be that mathematical rigour is required, in the most original and innovative theories, only at a certain point in their development, and only once the fundamental concepts are sufficiently clarified so that all that remains to be done is a work of fine-tuning. In any case, even these deviations from the conception of a mathematics that would be perfect in its coherence and the linearity of its progress can be analysed in a transcendental context, where they take all their meaning and legitimacy. It had to be pointed out, as we meet here for the first time with Gauss this idea of rigour.

If the seeds of a movement of transformation—of mathematics in general and algebra in particular—can be found in his work, it must be said that his case is exemplary but far from being isolated. The list would be long of his contemporaries who reached on such or such point intuitions, if not similar, at least of similar scope to his own. It will be enough to quote Lagrange!<sup>6</sup> Since we cannot give each of them their due, we will retain from the birth process of modern algebra only a few names. So we shall move directly, immediately after Gauss, to Galois. Evariste Galois,<sup>7</sup> haloed by a resolute republican commitment and a death in a duel before the age of twenty-one, is one of those rare mathematicians with a romantic figure

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<sup>3</sup> Augustin-Louis Cauchy (1789–1857). French mathematician, one of the masters of analysis and rigour.

<sup>4</sup> Niels Henrik Abel (1802–1829). Norwegian mathematician; one of the legislators of the analysis with Gauss and Cauchy.

<sup>5</sup> Thus the work of Edward Witten (Fields Medal 1990) and Maxim Kontsevich (Fields Medal 1998).

<sup>6</sup> Joseph Louis Lagrange (1736–1813). Franco-Italian mathematician. Celebrated, among other things, for his work in mechanics.

<sup>7</sup> Evariste Galois (1811–1832). French mathematician, craftsman of the theory of groups and algebraic structures.

and destiny. His contributions deal with the resolution of algebraic equations, the biggest problem algebra has had to face in its history.

The notion of a commutative group was implicit in some of Gauss's work; the same would apply to Lagrange or Abel. The more general idea of group<sup>8</sup> (without the condition of commutativity) would be found in the same way in Ruffini<sup>9</sup> or Cauchy, in questions related to problems of solvability of polynomial equations. One of Galois' genius traits is to have been able to glimpse the scope of the concept, opening up perspectives and a new horizon of applicability. Thus, Galois is often called the first modern algebraist, because he was the first to conceive the organizing power of general algebraic concepts, often obtained by abstraction from familiar situations. The identification of an algebraic structure can, on its own, radically transform the very nature of a problem and its meaning. Galois was able to move the terms and the formulation of the most burning questions in the algebra of his time by inserting them into a new theory. Deciding on the solvability of polynomial equations will refer after him to the study of similar problems for the corresponding groups. Galois' ideas are amply described in mathematical historiography; we will only insist on their methodological scope and their emblematic character.

The process of the Galoisian discoveries is characteristic of an eidetic method of investigation and creation—the same one that is at the center of Husserl's idealism, among others in the *Cartesian Meditations* [49]. The idea of the method is to treat as autonomous objects of study the intuitions and structures that appear in attempts to solve a problem. Galois is confronted with several phenomena, characteristic of the behaviour of the solutions of a polynomial equation. In order to abstract from them a new algebraic theory, it was first necessary to understand the regularities of this behaviour, i.e. to identify typical properties, common to the various polynomial equations. Mathematical creation is first born from observation, at least when authentic discoveries are involved. By observation, in the case of Galois we understand a familiarity with polynomials, a methodical study of their properties and a sustained attention to the permanence of certain phenomena during the transition from one type of polynomial to another.

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<sup>8</sup> A group is a set  $G$  with a composition law  $f$  with the following properties:

1.  $f(f(x, y), z) = f(x, f(y, z))$  for all  $x, y, z$  in  $G$  (associativity)
2. There is an element  $e$  of  $G$  (called the neutral element), such that:  $f(e, x) = f(x, e) = x$  for all  $x$  in  $G$
3. For all  $x$  in  $G$ , there is an element  $y$  of  $G$  (called the inverse of  $x$ ) such that  $f(x, y) = f(y, x) = e$ .

A group is said to be commutative if, in addition, for all  $x$  and  $y$  in  $G$ ,  $f(x, y) = f(y, x)$ .

<sup>9</sup> Paolo Ruffini (1765–1822). Italian mathematician, the first to have demonstrated the non-resolvability by radicals of a general polynomial equation of degree 5 (i.e. the impossibility of expressing its roots from the coefficients using only the operations of addition, subtraction, multiplication, division and taking a root). Ruffini introduces in his work several of the fundamental concepts of the theory of permutation groups.



Algebraic regularities are rarely self-evident. Those of the roots of polynomials would remain hidden from the mathematician who, unaware of Galois' work, would look into these problems, hoping to solve them without a certain humility and without any familiarisation with the object of his studies. Once a theory is already in place, it is of course possible to seek to increase it with new results, obtained in a purely formal way from technical problems raised by the mechanical development of the theory. Most of the published mathematical results are of this order. They are indispensable to the proper functioning of constituted science and allow the rational organisation of the corpus. On the other hand, they cannot claim greatness, no matter how virtuoso they are. We will come back to these questions in more detail, based on the exemplary testimony of Alexandre Grothendieck.

For the moment, let us stick to the discoveries of Galois. At the origin, therefore, there is in them the detection of typical forms in a series of phenomena. This corresponds to perhaps the most exciting moment in the process of creating algebraic concepts: the moment when the regularity of the phenomena reveals a hidden intelligibility. Work allows us to identify little by little the reasons for this regularity—the causes, one would like to say if causality were not an inoperative concept in mathematics. This is the second moment, when a precise form is finally given to the hidden structures, the one that allows us to go from intuition to the concept. It is the most authentically transcendental moment of mathematical creation. In the case of Galois, the concept thus identified is that of group (and other associated concepts, such as that of normal subgroup). It crystallizes all the insights that had been previously obtained. However, at this point in the mathematical work, no quantitative progress has been made. The problems remain, but from now on a framework is fixed, within which they are organized and take on a new meaning. Recently, the demonstration of Fermat's theorem has resulted, almost entirely, from such a change of framework! Once original organising concepts have been identified, what remains to be done is to work with these new objects. With a bit of luck, their study will enable us to solve the initial questions, which will prove easier to solve once they have been transcribed into the eidetic horizon that has just been created.

In short, Galois' work is exemplary of a new method. It consists in extracting from the frequentation of mathematical objects structuring concepts, which are objects of a higher order in abstraction and delimit a field of mathematical research. In Galois—and with him all those of his predecessors who contributed to the building of the theory—, group theory is thus established as a field of investigation. These entities, the groups, which appeared in the course of various problems, from Gauss's arithmetic to Galois' work, are new mathematical objects in their own right. Their discovery opens the way to a more resolutely abstract and unified conception of algebra, since the example of groups will show that the same concept is likely to intervene in diverse and, apparently at least, distant mathematical situations.

A word before passing on to the successors of Galois. Anything in the field of mathematical history that involves abrupt distinctions and claims to delimit breaks, spectacular innovations, must be suspect, because it obscures the generally long work of maturing concepts. Galois was a genius, but he inherited the efforts of

generations of mathematicians, from the Italian algebraists of the Renaissance to Gauss, Lagrange, Cauchy, Ruffini. . . The remark applies both to the mathematical concepts and techniques introduced by Galois and to the spirit (style) in which they were conceived. As far as the eidetic method is concerned, it has always haunted mathematics as a power that only needed to be put into action and, even if it was not until the nineteenth century that it took shape, it would probably be possible to find traces of it in all eras. If we have chosen to retain Galois as the exemplary representative of the entry into the modern era in mathematics it is because of the magnitude of his contribution, but also because of his historical situation and his role as a trigger in the renewal of algebraic methods in the nineteenth century. In terms of content, it will be remembered that the Galoisian method has its origins in the ability of the mind to abstract phenomena from typical forms; as such it is found in all forms of organised abstract thought. It is in this capacity that it intervenes in a very general form in transcendental idealism.

After Galois, Grassmann, Cayley and Hamilton,<sup>10</sup> the great name in nineteenth century algebra is, chronologically, without a doubt Richard Dedekind. He was the last of Gauss's pupils in Göttingen. His first works were devoted to analysis, and it was the preparation of courses of analysis that motivated his famous re-elaboration of the theory of irrationals around the notion of "cut". The thought of Dedekind and his very algebraic approach to arithmetic problems and problems of foundations were decisive in the constitution of modern mathematics. It could be said that many of the fundamental orientations of modern mathematics, in all their power as well as in their sometimes debatable aspects, can be traced back to him, without whom they would not have had the scope that we know them to have. Kronecker<sup>11</sup> in particular, whose arithmetic results rival in magnitude with those of Dedekind, could have taken the algebra in a much more algorithmic direction. The spirit of the discipline would have changed: it is well known today that the priority long given to formal structures is only one of the possible ways of developing algebra and, more generally, mathematics, which can also be approached through problems of effectiveness or computability. From this point of view, the development of computer science has represented a turning point, rehabilitating a certain number of approaches of an algorithmic or combinatorial type, and thus partially calling into question the algebraic-formal revolution which was born with Galois and Dedekind.

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<sup>10</sup> Hermann Grassmann (1809–1877), Arthur Cayley (1821–1895), William Rowan Hamilton (1805–1865). Founders, among others, of modern linear and multilinear algebra. Despite his genius, Grassmann's work had little resonance in the mathematical community of his day. The work of Cayley and Hamilton is in line with the symbolic development of algebra. We insist on the work of Dedekind because it marks the beginning of the German algebraic school, whose study is the richest in epistemological teachings, from our point of view.

<sup>11</sup> Leopold Kronecker (1823–1891). German mathematician, one of the great arithmeticians of the nineteenth century.

After his beginnings in analysis, the arrival of Dirichlet<sup>12</sup> in Göttingen in 1855 reoriented Dedekind towards the study of algebra, in particular through the work of Abel and Galois. Beyond this disciplinary influence, Dirichlet undoubtedly also intervened in Dedekind's tendency to seek, as a general rule, conceptual rather than computational solutions. In Dedekind's work, the Galoisian eidetic method took indeed its full extent. His work can be characterized as the search for organizing concepts aiming at isolating in each problem the fundamental idea underlying it (its algebraic core). The method is indeed the same as the one that guided Galois: it is a question of identifying each time the concepts or transcendental structures underlying the given problem, structures that emerge little by little and take shape through the frequentation of the considered objects. The novelty, in Dedekind, is in the systematic character and the universality of its application. However, even with him, the method remained implicit, and it was put into practice without being really theorized. It was only at the beginning of the twentieth century and with works such as van der Waerden's *Moderne Algebra* [82], which served as a modernist manifesto, that modern algebra was thought of as such and identified in all its scope.

The best known of Dedekind's contributions concerns the foundations of mathematics. In two famous works, *Stetigkeit und irrationale Zahlen* and *Was sind und was sollen die Zahlen* [21], Dedekind redefines the irrationals and the natural numbers. In both cases, he uses an idea that runs through his entire mathematical work: to substitute a calculation on the elements of a set with a calculation on its subsets. The idea, which has become familiar today, is bolder than it seems. It supposes that one treats as objects of calculation what was until then only collections of objects, thus allowing a transfer of intuitions and methods. In the case of irrationals, Eudoxus had already envisaged to build these numbers from the rational numbers by a very original calculation. The skill of Eudoxus is, by the way, in the image of the creative vein of Hellenic mathematics, far superior to the caricatured image which is still sometimes given of it. Using order relations on quantities, Eudoxus determines a ratio of quantities  $X/Y$  (which can be irrational) by dividing the ratios of rational numbers  $m/n$  into two classes. This division makes it possible to set up rigorous calculation methods, such as the exhaustion method.

Dedekind's method is very close to that of Eudoxus, although it differs from it by its conceptual abstraction and by the mode of definition of the objects used. Eudoxus reasoned without questioning the existence of the irrationals and, even though he had all the technical tools to do so, refrained from giving an explicit definition of them. The continuity of the set of reals was not really established, even if it resulted implicitly from the properties of the concept of geometrical magnitude. It is the weak point, from a modern point of view, of the Eudoxian theory, which

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<sup>12</sup> Johann Peter Gustav Lejeune Dirichlet (1805–1859). German mathematician; his family was of Belgian origin, hence his surname (“Le jeune de Richelet”). According to Felix Klein, “Dirichlet liked to make things clear to himself in an intuitive form; with this he was able to analyse fundamental questions with precision and avoided long calculations whenever possible”. In addition to his work on number theory, Fourier series and harmonic functions, Dirichlet influenced mathematics through his style of thinking, which left its mark on Dedekind and Riemann.

admits that magnitudes form a well determined set without trying to specify its nature. Dedekind reverses the perspectives. The method of cuts, which was used by Eudoxus to characterize an irrational, becomes for him the process by which they are defined. Henceforth an irrational will be identified with a cut, i.e. with the division that it determines of the set of rationals into two subsets—those smaller and those larger than the irrational under consideration. What was until him a property of the real numbers, becomes the principle of their definition. The changeover is not very significant from a strictly technical point of view; any mathematician with a little practice recognizes at a glance an identity of essence between the work of Eudoxus and that of Dedekind (at least if one neglects the problem of continuity, which in any case could be treated as well in the formalism of Eudoxus as in that of Dedekind). Yet the passage from one to the other is anything but immediate in that it presupposes a change of attitude towards the manipulated concepts and their logical status. The method of modifying the status of the properties of mathematical objects by giving them autonomy and their own content has in fact become so common today that it has become difficult to appreciate its depth in retrospect. In the same way, it is now difficult to conceive Euclidean geometry without the use of coordinate systems, at the risk of anachronisms!

Dedekind's other discovery in terms of foundations is the subject of *Was sind und was sollen die Zahlen*. Dedekind's problematic was "in the air" at that time, as many people were dealing with elementary arithmetic, and definitions of integers flourished in the literature of the time, from Weierstrass to Cantor. Dedekind immediately asserts an idealism often described as logicist, at the price of an unscrupulous reading of his work.

By defining arithmetic (algebra, analysis) only as a part of logic, I affirm that I consider the concept of number to be totally independent of representations or intuitions of space and time, and that I see it rather as an immediate emanation of the pure laws of thought. [21]

This excerpt from *Was sind und was sollen die Zahlen* clearly shows that the term 'logic' is to be taken in a completely different light than that which would be spontaneously attributed to it in a modern text. As with the first Frege, there is nothing to suggest that the term 'logic' in Dedekind is devoid of a classical philosophical component. Dedekind's vocabulary would tend to confirm this hypothesis, since the "laws of thought" lend themselves to many interpretations and cannot easily be reduced to their axiomatic-formal component. It is also necessary to note the fierce opposition which is asserted in the text to any foundation of the concept of number in any transcendental aesthetic whatsoever. Here again, similar traits are found in Frege and testify to a rejection of a part of criticism which can easily be isolated from the rest of Kantian theory. In general, from Dedekind's work emerge more epistemological problems than settled theories. Thus, his thought never strayed into the logicist impasse. The abusive identification between analytic thought, mathematics and formalism will only come after Frege.

Dedekind's definition of integers relies on what is now called (elementary) set theory. It is based on the notions of application and set, and on the less classical notion of chain, which makes it possible to establish induction. An example will

illustrate his method. A characteristic of infinite sets is that they can be put in a one-to-one correspondence with one of their strict subsets. Dedekind transforms this property into a definition of infinite sets and, by the negative, of finite sets. Here again, he succeeds in isolating in the properties of a system certain components, representative enough to allow the theory to pass through them to a higher level of coherence and abstraction.

The core of Dedekind's contribution to the birth of modern mathematics, however, is not based on these fundamental questions, but on his results in algebraic number theory. It is there in particular that he developed the language of ideals, fields or modules. The elaboration of his manuscripts on the theory of ideals extends from 1871 to 1894; the theory moreover evolved towards an always greater level of abstraction—and of eidetic clarity—between these dates. The starting point is the fundamental theorem of arithmetic: the decomposition of a natural integer as a product of prime numbers. Gauss the first, after a few unfinished attempts by the Basel mathematician Euler, had generalised it to what are now called Gauss integers.<sup>13</sup> Kummer<sup>14</sup> then extended Gauss's results to cyclotomic integers.<sup>15</sup> Kummer's results are based on the rather ambiguous notion of "prime ideal numbers". His approach is conceptually very bold since he manipulates numbers whose definition remains implicit. Kummer's genius is precisely to sense the existence of hidden structures and to reveal their empirical manifestations, to exhibit laws whose essence remains temporarily hidden—somewhat in the way that "imaginary" numbers have been used for a long time without their existence being considered really certain.

The transition from Kummer's prime ideal number theory to Dedekind's viewpoint is typical of the way modern mathematics works. Theories are not born out of thin air, and often a number of results and objects that have appeared in the course of calculations are available, but the ultimate reasons for their existence are uncertain, as if they lacked architectural support. These phenomena are not a matter of logic, or even of strict rationality. It is sometimes obvious that something is missing for the intelligibility of the phenomena studied to be complete, and when the missing construction or idea ever surfaces, it is immediately recognised as the element that was missing, even though there was no indication as to its precise nature until then. A mathematical theory can therefore, even if it is coherent and logically satisfactory, give a more or less clear-cut feeling of incompleteness, as if the latticework of its concepts and results indicated to the mathematician the existence of an idea which is still missing but necessary to its perfection.

The discovery of Dedekind comes from the feeling of a gap to be filled, an eidetic deficit in Kummer's work. The most original of the concepts that he introduces

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<sup>13</sup> The Gauss integers are the complex numbers that have integer coordinates when considered as points on the plane.

<sup>14</sup> Ernst Eduard Kummer (1810–1893). One of the founders of the algebraic theory of numbers.

<sup>15</sup> These are complex numbers defined as integer-coefficient linear combinations of complex numbers that are roots of unity, i.e. one of whose powers is the unit.

allows the advantageous substitution of ideal numbers; it is the notion of ideal.<sup>16</sup> One of its characteristics—we already noticed that it runs through the works of Dedekind—is that it allows to make calculations no longer on elements of a set but on sets themselves, of which one can define for example sums or products. Calculations with numbers in the traditional way are advantageously replaced by calculations on collections of objects. The same process appears in other domains than the algebraic theory of numbers: all modern geometry rests on an analogous principle of substitution to the points of space of algebraic entities which serve as substitutes. Here again, the notion of ideal is capital and the historical role of Dedekind is decisive.

Ideals are only one of the concepts introduced by Dedekind, but they illustrate well the approach to algebraic questions that emerges with his work: the search for structuring concepts, the displacement of classical problems towards objects of a higher order, both in abstraction and in generality. Yet this shift remains partial. The concepts introduced often keep an instrumental content and are not ordered within an organized set of “algebraic structures”. Until recently, high school mathematics classes taught that groups, rings, fields, vector spaces and other similar entities form a coherent whole, the passage from one notion to another being obtained by adding further axioms.<sup>17</sup> In Dedekind, this idea of a hierarchy of organic concepts of algebra is still lacking.

It was only in the twenties of the twentieth century that it would become clearer, leading to modern mathematics and a vision of the discipline, its logic and its objectives, which were able to penetrate for a time into French secondary education. The latter, under the effect of mass education, demagogic impulses and various misunderstandings, had then to regress and abandon most of its conceptual ambitions, to retain of mathematics only a few insipid and little formative residues. This is another story, not very glorious, and which continues to be played out today. The reform of modern mathematics wanted to introduce into teaching the eidetic clarity which had been gradually established in mathematics. The fundamental concepts of algebra, for example, such as those developed by Dedekind, have a unifying and clarifying role; they make it possible to synthesize the whole experience of generations of mathematicians. They retain only the essential ideas and avoid cluttering the mind with a host of heterogeneous notions. This effect is obtained at the price of an undeniable effort of abstraction, and the risk is great that the pupils, confronted with too esoteric mind games, do not understand either their meaning or their objectives. So there was a huge work to be done to make people understand that this reform was the result of a legitimate will and a great intellectual ambition. Perhaps this work had to be accompanied in the high schools by some lessons of history of thought, or even epistemology? But perhaps also these two disciplines were not in their turn, at that moment in history, at the level

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<sup>16</sup> An ideal of a ring is subgroup stable under products with the ring elements.

<sup>17</sup> Thus a ring is a group with an additional composition law, subject to certain conditions of compatibility with the group structure.

of such a project? Anyway, the failure is felt more and more every day and the teaching of mathematics has for some time now not stopped giving up a little more its conceptual wishes.

The responsibility is shared. Although he vigorously defends himself against it, the somewhat vain intransigence with which a confirmed and influential mathematician like Dieudonné<sup>18</sup> could claim to separate the wheat from the chaff when it comes to mathematical truths has undoubtedly not helped the debate. Each of his judgements in favour of the structuralist programme,<sup>19</sup> whether in history or mathematical thought, breathes scientific culture and intelligence of content, but terribly lacks openness to opposing arguments. In fact, according to him, no debate is even possible: there is on the one hand the unanimity of an elite—which he claimed to represent, sometimes abusively—and on the other “those who have not received the gift of an inventive imagination”. Let us quote an extract from his texts on Bourbaki:

In other words, no one can understand or criticise the choices made by Bourbaki unless they have a solid and extensive culture in many mathematical theories, both classical and more recent. [27]

“Mathematics is the opposite of a democracy”, he argues in the same text. This type of haughty attitude could not fail to catalyse oppositions to structural thought which, from a certain moment on in recent history, was no longer defended by a living epistemological project. Dieudonné is profoundly wrong. The community of scholars is a Republic, which certainly has exceptional citizens, but admits only one law: that of reason. Even if an epistemological decision is right, it must be open to public discussion. Arguments of authority are never more than a poor substitute for a well-founded argument. Galileo already knew this, as he remarked in his *Dialogue on the Two Great Systems of the World* [32] how much medieval Scholastics misused the authority of Aristotle: a great thought, such as that of the Stagirite, can only benefit from being debated, and is betrayed when it becomes dogma. One of our objectives is to revive the debate on mathematical thought, far from any dogmatism, in particular by returning without prejudices on the structuralist heritage.

A few remarks will serve as a conclusion to this chapter on the origins of modern algebraic thinking. We have seen in Dedekind the premises of modern set theory. We will speak very little about it in the following, in a deliberate way. Indeed, the effective interest of logical questions for mathematical thinking, although for long a hot topic in the philosophy of mathematics, should not be overestimated. It would be distorting the perspective to give it too much importance. Instead, it is necessary to specify the way in which mathematical thought reorganised itself by freeing itself at the end of the nineteenth century from the grip of the Kantian

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<sup>18</sup> [Translation’s note] One of the Bourbaki group founding members, Dieudonné was a very popular and influential figure in French mathematics. More on him and Bourbaki in forthcoming chapters.

<sup>19</sup> The precise content of this programme will be described in Chap. 6; until then we will continue to use the concepts of mathematical “structure” and “structuralism” quite freely.

tradition of aesthetics—that is to say, concretely, from the role of the intuition of space and time.

The project had already been formulated by Dedekind, but it is with Frege that everything was decided. Frege's work marks a decisive turning point in the overcoming of Kantism that accompanies the implementation of the various axiomatizations of geometry and arithmetic. For a long time, history has only retained certain aspects of it, essentially the logicist project or the idea of a formula language of pure thought,<sup>20</sup> which can be partly found in the programme of modern mathematical logic. The fact that Russell was the one who revealed Frege to the general public undoubtedly played a role in the rather narrow image we often have of his thinking. All its richness is little by little rediscovered through the studies in analytic philosophy, which has for a long time taken the measure of Frege's ideas on semantics, such as the distinction between meaning and reference (*Sinn und Bedeutung*) of an expression. However, despite the exemplary work accomplished by all his commentators, it is not clear that the scope of his work has been well understood with regard to the fundamental principles of mathematical thought, which are still too often judged by the yardstick of philosophies or theories too exclusively conditioned by logic, syntax or grammar.

A single fact will suffice to give the measure of its richness and complexity. Even though he is one of the fathers of logicism, his later works deviated completely from it and he ended up turning to geometric ideas to establish the foundations of arithmetic. Frege is probably the thinker who had the greatest sensitivity to the problem of the meaning of concepts, and it is easy to measure what this can have meant at the very moment when an axiomatic-formal vision of mathematical thought was being developed around Hilbert. In short, he is one of those rare authors whose reading convinces us that they go to the essential and prefer the search for truth to easy certainties.<sup>21</sup>

In his *Foundations of Arithmetic* of 1884 [31], Frege sets up the construction of the concept of number via finite sets, which he understands explicitly as the decisive passage from Kantian theory to an “analytic” theory of mathematical judgements. Consciously or not, all axiomatization works presuppose that this Fregean reduction of mathematics to an analytic science, and as such susceptible of formal descriptions, has been accomplished at the philosophical level. However, Frege's place is atypical in that the vocabulary and philosophical references in his *Foundations of Arithmetic* are still explicitly Kantian, and Frege conceives his work more as an overcoming than as a questioning of Kantism.

A superficial analysis suggests that he does nothing more than a set theoretic axiomatisation of numeration, essentially of the kind presented in modern textbooks.

<sup>20</sup> Idea developed in his 1879 book, *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (Concept Notation: A formula language of pure thought, modelled upon that of arithmetic).

<sup>21</sup> Husserl is also one of them, and one can only contrast them with the brilliant but often histrionic epistemologies of their successors.



He describes a number as an equivalence class of finite sets for the equivalence relation defined by bijections, so that the only concept required to found arithmetic seems to be that of equivalence relation, which is a matter of general logic. The mathematical truths would thus be reconductible to logic.

Yet Frege's analyses go far beyond this logicist reading of his work. We need to return to the vocabulary he uses to understand the meaning of the analytic turn he brings to mathematical philosophy. For him, a number is first of all "the extension of a concept" (der Umfang eines Begriffes). To number is to state an objective determination of a concept. This definition is therefore not purely formal but also intentional. To define the concept of number is to have in view a field of situations where it makes sense and where intuitions are likely to fill it, i.e. where objects can correspond to it. Defining a number thus seems to be at the origin a more synthetic than analytic act since its concept is abstracted from a relation between objects.<sup>22</sup> The fact that arithmetic is likely to be codified in a symbolic writing does not change this.

If one reads Frege by modern standards, there seems to be a contradiction in his assertions. On the one hand he explicitly claims the analytical and purely logical character of his set theoretic axiomatisation of the number, on the other he constructs the corresponding concept in terms which refer de facto to a synthetic mode of constitution. Frege is fully aware of this paradox and takes it into account. The solution he gives makes it possible to reconcile intuition and formalism and, as such, deserves attention, for the end of all the difficulties of mathematical thought and all its aporias lies in the possibility of such a reconciliation.

The Fregean idea is simple and goes in the direction of transcendental idealism, for which we want to plead, by inserting it in an analytical-formal problematic. We find an echo of this in some of Lakatos's works and his defence of a critical rationalism. In two words, it is not because the construction of the concept of number requires the intuition of sets—thought as elements of equivalence classes—that this concept does not have an aprioric necessity and is not valid as a pure form of understanding. It is not because the use of intuition is necessary to our practice of mathematics that the corresponding concepts do not have an unconditional validity. In Fregean terms, echoing his distinction between meaning and reference: "the truth of a statement does not coincide with what is thought in it". A theorem has an objective content which will be reconductible to analysis, to formal languages or to a suitable axiomatic. But it also has a meaning. Alongside the logic of formal truth, a logic of meaning appears, a logic of "what is thought", a logic of the contents of consciousness, which has its own legitimacy and its own function in the edifice of mathematics. Even if mathematical statements are valid a priori and independently of any experience, the fact remains that, in order for us to simply conceive them, we must first of all realise them in an intuition. And it doesn't matter if this intuition is questionable!

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<sup>22</sup> [Translation note] On these questions, see F. Patras, *The Essence of Numbers*, Springer, 2020.

Frege does not go that far, but his work lends itself readily to such developments. The ultimate reversal of Frege's thinking, his renewed interest in geometry, is a sign that mathematical philosophy must never stop at the comfort of logical certainties. Gödel's work leads in its own way to the same conclusions.

## Chapter 5

# Axioms and Intuitions



**Abstract** Originally thought of as a way to build foundations, the axiomatic method was progressively used more and more to organise and expand theories, their scope and boundaries. Despite their formal content and nature, the role of axioms in mathematics cannot be understood without taking into account mathematical practice and intuition.

**Keywords** Axiom · Intuition · Intuitionism · Formalism · Hilbert · Weyl

Among the founders of modern scientific thought, a figure stands out between mathematicians and logicians. It is David Hilbert, whose ideas changed the course and nature of mathematical epistemology. In 1899 Hilbert published his famous *Foundations of Geometry* [45], which definitively associated his name with the axiomatic enterprise—or, more precisely, with its renaissance, for the idea of axiomatic thought does not arise with our century. The Aristotelian *Organon* had already established its foundations and methods, and the geometry of Euclid had come to give ample legitimacy to its ideas. Hilbert, however, changed the perspectives of Aristotelianism. In technical terms, his geometry surpasses Euclid's by its logical architecture, but this is almost a point of detail: pure technique hardly carries any meaning, and if this had been the only Hilbertian achievement, his contribution would not have had such brilliance.

What made the *Foundations of Geometry* so successful was first of all that Hilbert dissociated axiomatic thought from natural thought. The objects of geometry, in the Hilbertian perspective, are founded in law and reason by their formal definition and independently of intuition. Research into the foundations of geometry was not new, but it had not previously taken such a radical philosophical turn. Roberval,<sup>1</sup> Wallis<sup>2</sup> or Leibniz had already tried to identify the breaches of the Euclidean approach, not to mention the numerous researches undertaken since antiquity on the postulate of

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<sup>1</sup> Gilles Personne de Roberval (1602–1675). French mathematician, inventor of the famous balance.

<sup>2</sup> John Wallis (1616–1703). English mathematician.

parallels and its logical independence from the other postulates of geometry. In the nineteenth century, research had focused on projective geometry and its fundamental structures. Two mathematicians were particularly well known in this field. Firstly, Felix Klein,<sup>3</sup> who remained famous for his work on the role of transformation groups in the characterisation of different geometries, ideas which were explained in his inaugural professorial address, known as the *Erlangen Programme* [59]. Klein's insights seem to date back to his travels to Paris, where he became familiar with Lagrange's ideas on group theory. Klein understood their universality and unifying potential, even in a geometrical context and in rather unexpected ways, which justified his fame and the historical importance of his ideas. For his part, Moritz Pasch<sup>4</sup> set out to give projective geometry an axiomatic basis. His work was to generate a current of interest in logical questions, among others through the Italian school and, in particular, through Giuseppe Peano, whose problematic proved in retrospect to be very close to that of Hilbert, at least in technical terms.

Interest in questions of axiomatic foundations was therefore already quite widespread when the latter tackled them, but, beyond his results, he was also distinguished by his epistemological views. Research on the foundations of geometry had long been based on a realistic conception of the objects of space. Peano had been concerned about the correlation between logical and geometrical meanings, but Hilbert's views were able to impose the idea of an absolute independence of axiomatic-formal terms with regard to empirical issues. Yet this formalistic conception is not as monolithic as it seems, and Hilbert was always able, as a mathematician, to temper his views and keep in mind the problems posed by formalism. The radicalisation of his views will only come rather late and will assert itself as a reaction to the "provocations" of Brouwerian intuitionism.<sup>5</sup> It will always remain anchored in a reflection on the essence of thought:

For this formula game is carried out according to certain definite rules, in which the *technique of our thinking* is expressed. These rules form a closed system that can be discovered and definitively stated. The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds. . . . (D. Hilbert, quoted in [74])

It was only after Hilbert that formalising thought ceased, in its most extreme assertions, to be concerned with the ontological questions that underlie any theory of knowledge worthy of the name. It is worth emphasising this point because Hilbert's work is, along with that of Poincaré, one of those demarcation lines in the history of thought where the theory of science denies part of its heritage in order to allow itself to be seduced by the most radical potentialities. The impression that emerges

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<sup>3</sup> Felix Klein (1849–1925). German mathematician, geometer.

<sup>4</sup> Moritz Pasch (1843–1930). German mathematician, geometer.

<sup>5</sup> Contrary to formalism and demonstration theory, intuitionism, developed by the Dutch mathematician L.E.J. Brouwer (1881–1966) emphasises the subjectivity of the mathematician and the necessary finiteness of the tools he may use for theoretical exploration, a finiteness that Hilbert only required at a metamathematical level to keep using infinitary mathematics (Cantor. . .).

from the mathematical practices of the last decades is that the reduction of the Hilbertian heritage to some of its formalist traits, whatever its legitimacy, has led to the sidelining of fundamental questions, such as that of the ontological significance of mathematical thought or that of its relationship to the natural sciences, a question to which Hilbert was particularly attached. His conceptual radicality, his excessive expectations regarding knowledge, his intellectual optimism, seem to have been transmuted into a tacit abandonment of many questions of the theory of knowledge. Mathematicians are now too often content to “do their job” honestly, without looking beyond the deductive straitjacket in which their thinking has chosen to be imprisoned, within axiomatic systems whose intrinsic limitations are nevertheless well known.

What was the meaning of his undertaking and what avenues does it continue to open for reflection today? We now know that not all the most radical solutions in terms of foundations can succeed, whether on the one hand an exacerbated formalism, which stumbles on the results of Gödel, or on the other hand a transcendental idealism with ontological anchoring which would not take into account the successes of the formal logic. By the end of his days, Hilbert had become fully aware of the difficulties that mathematical thinking could not fail to face. The latter is full of ambitions that are equally legitimate but difficult to reconcile, as it is experiencing a tension between its vocation to describe reality and the search for unconditional coherence. Unity must however be possible, at the end of a dialectical synthesis of the intuitive and axiomatic-formal moments of thought, but the outcomes are most uncertain. Up to now all attempts to reconcile the formal and the transcendental have been in vain. Yet the future of mathematical science is at stake, and its legitimacy is at the price of such a synthesis. Hilbert would probably not have doubted its possibility—even if it would have meant, for him, favouring more technical options than those we would tend to prefer here.

Geometry, like arithmetic, needs only a few, simple principles to be properly edified. These principles are called the axioms of geometry. The determination of the axioms of geometry and the study of their interdependence is a task which, since Euclid, has been dealt with in many excellent treatises of mathematical literature. This study boils down to the logical analysis of our spatial intuition. (D. Hilbert [45])

The significance of the initial Hilbertian project becomes apparent if one adds to these first lines of the treatise on the *Foundations of Geometry* the fact that they are preceded by a quotation from the *Critique of Pure Reason*. Thought progresses only little by little in the abstraction and understanding of its presuppositions. It rises from intuition to pure concepts, and, as was already the case with Frege, this purity is susceptible to analytic treatment. This is the essence of the axiomatic method, which allows us to translate the system of our intuitions into a rigorous form. It is first of all a process of abstraction starting from sensitive data. Its independence with regard to sensibility is thus relative. The results deducible analytically from the axioms are logically independent of any intuition, but they do not cease however to refer as at their origin to the system of our perceptions. Once again, we recognise in the watermark of this description the features of transcendental idealism, except

that it is no longer a question of studying the pure concepts of understanding or other similar objects, but rather the truths of a formal discourse, if not of a formal language.

The system of axioms of geometry is divided by Hilbert into five subsystems which correspond to distinct types of eidetic intuitions. Thus, although these axioms are intended to deal with entities potentially devoid of intuitive meaning, he never ceases to subordinate them to the intuitions that correspond to them, and thus to a legality that is that of ordinary thought. To put it another way, intuitive interpretation always emerges in mathematical writing as a legitimate possibility, right to the heart of axiomatisation. It would be both vain and awkward to renounce it. Thus, Hilbert never stopped refusing a blind formalism that would originate its own questions. Axiomatics is at the service of mathematical thought, which it brings to its highest degree of clarity and assurance; in no way does it encompass it or substitute for it. Mathematics must remain master of its object. The bad process of reducing it to a vast tautology is not conceivable in the Hilbertian theory of science, which is too well informed of the real meanings underlying scientific work to be misled by a mechanical use of reasoning. The mathematician's intuition always serves as a safeguard for Hilbert, even in his most daring projects to reduce mathematics to a pure game of forms, projects which, moreover, in his works do not aim at dispossessing mathematics of its meaning, but at guaranteeing its validity—which is quite different, even if formalist research leads, when it unfolds, to temporarily putting the meaning of mathematical statements in brackets. His verb would sometimes go beyond his thinking and allow the most extreme interpretations.

Contemporary scholarship, which has gone into the details of Hilbertian manuscripts and works, bears abundant witness to his reticence towards a reductive vision of the development of science:

The edifice of knowledge is not erected like a house, where the master walls are first built before the living rooms are constructed and fitted out; science prefers to build living spaces as quickly as possible. It is only then, when it appears that here and there the fragile foundations of the building are not sufficient, that it goes ahead to support and reinforce them. There is nothing lacking there, and this is indeed the proper and healthy way in which it must develop (D. Hilbert, quoted in [73]).

The idea is in line with Kant's quotation in the *Foundations of Geometry* that we have had the opportunity to mention: all human knowledge begins with perception and proceeds from there to build its concepts. The difficulties of Hilbertian thought lie in the tension thus created between intuition and pure forms, between axiomatic thought and human experience. They are still reflected in the opposition between the genetic method and the axiomatic method in the definition of mathematical concepts.

This opposition is found in the philosophy of knowledge, which classically distinguishes between genetic and formal logic. In two words, the genetic method is the one by which we construct notions, in the very mode according to which they were first apprehended by the mind. The definition follows the historical or natural progress of the mind in its mode of elaboration of concepts. The genetic method is also the pedagogical method which appeals most to the intuition, by anchoring in it

the most abstract ideas. It is the method of Platonic discourse where the interlocutors gradually become acquainted with an idea by learning to guess its contours. As for the axiomatic method, we know it well enough: it consists in defining concepts from the outset in their pure and ideal form and in all their sophistication. The student is confronted with a perfectly constituted and coherent science, which appears to him in its greatest degree of perfection. He immediately has the most powerful conceptual tools at his disposal, without having to pay the price and by virtually skipping steps. On the other hand, he will lack the corresponding intuitions, like those students who think they know this or that abstract theory but would be at a loss to say what it applies to apart from a few formal and agreed examples.

In *David Hilbert and his Mathematical Work* [87], a posthumous account of Hilbert's work, Hermann Weyl analyses his mathematical and logical contributions. It is a particularly important testimony, due to the proximity of the two men. The latter's attraction to Brouwer's ideas contributed greatly to Hilbert's reaction against intuitionism, an unmistakable sign of the esteem and credit he had for Weyl's ideas, who had been his pupil and was now an adversary. Weyl's view of Hilbert's logicism is nuanced, and its relevance seems today to be beyond doubt. It must be said, moreover, that Weyl's philosophical and mathematical work, although not always easy to read, is imbued with an intuitive sensitivity, which goes not without procuring aesthetic pleasure. This is an extremely rare sensation, found only on very few occasions. Therefore his judgment of the Hilbertian writing in his *Foundations of Geometry* is particularly significant:

With respect to the economy of concepts, Hilbert is more conservative than the Italians: quite deliberately he clings to the Euclidean tradition with its three classes of undefined elements, points, lines, planes, and its relations of incidence, order and congruence of segments and angles. This gives his book a peculiar charm, as if one looked into a face thoroughly familiar and yet sublimely transfigured. [87]

In other words, the Hilbertian mode of exposition deliberately aims to place axiomatics in the context where it makes sense, where it is best fitted into tradition. Axiomatics is the search for additional intelligibility, and it would be absurd to strip it of all reference to intuition, since this contributes to increasing its scope and clarity. What is most striking today in these debates surrounding the work of Hilbert is that a number of questions which have encumbered mathematical pedagogy long after his death had already been clarified and almost settled. Worse still, some problems of which Hilbert or Weyl were aware have been lost sight of, even though they had been formulated at the beginning of the last century. They are still waiting for attention.

Hilbert has always affirmed his confidence in the axiomatic method. On this point, the question was decided: this is the method that a theory must reach when it matures. But the fundamental problems are far from being limited to this. There are a number of logical questions to be decided in order to arrive at a coherent theory of science, and Hilbert points out the questions of decidability or completeness. But other questions go beyond the logical framework: in 1917, during a conference at the Swiss Mathematical Society, he thus evoked the problem of defining a standard

of simplicity for a mathematical proof or that of the relation between content and formalism. All these problems have more or less disappeared today from the framework of mathematical philosophy. Beyond the rather particular context of theoretical computer science, the few attempts to link scientific forms and contents which have been tried have mainly concerned physics, and their transplantation into a mathematical context, when it has taken place, has not led to satisfactory results.

There is, however, a philosophical technique whose ambition is to confront precisely these questions: phenomenology, of which Hermann Weyl has sometimes been one of the spokespersons. The case of Weyl is particularly interesting, in that he was one of the inventors of modern mathematics by de facto promoting the idea of structure, even in non-algebraic contexts, as in his book on *The Concept of a Riemann Surface* [86]. His works seem to pursue an ideal of balance between forms and intuitions, both of which are intended to support and complement each other in the service of a multifaceted and universally oriented thinking. As in Hilbert's work, axiomatic rigour is put into practice, but it only appears as the final moment of the movement of thought: the moment when the notions finish ordering themselves into a system. Two facts, among many others, bear witness to the way in which the mathematicians of the generation after Weyl's chose to ignore questions of the theory of knowledge, when they did not deliberately distort the thinking of their predecessors in these matters. In a notice on Weyl's work published in 1957, two years after his death, by C. Chevalley<sup>6</sup> and A. Weil,<sup>7</sup> two of the leading representatives of the structuralist movement, no mention is made of Weyl's philosophical research, except on the margins and in a rather irreverent manner, as if these were unflattering side-effects of his character that one would be well advised to forget. The thing is more serious than it seems: whatever Chevalley and Weil's view of intuitionism and of the epistemological problems that occupied Weyl, their deliberate concealment cannot fail to shock in retrospect. Beyond the notice itself, all this is characteristic of a methodological choice common to their generation: "serious" mathematicians would not have to concern themselves with philosophy, except perhaps privately and without it interfering with their mathematical writing. This is obviously an illegitimate choice, because all forms of mathematics are governed by epistemological options which, although they remain tacit, are no less decisive in the thematic and programmatic orientation of the work. For all that, this harmful choice, whose disastrous consequences have not ceased to be felt, has been accepted almost unanimously. Let us listen to Chevalley and Weil :

A Proteus, who constantly transforms himself to evade the blows of the opponent, and only becomes himself again after the final triumph: this is the impression that Hermann Weyl often leaves us. Didn't he go so far, pushed by the milieu no doubt, by the occasion, but also by "the anxiety of his genius", as to become a logician, a physicist, a philosopher?

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<sup>6</sup> Claude Chevalley (1909–1984). French mathematician, group theorist and active member of Bourbaki.

<sup>7</sup> André Weil (1906–1998). French mathematician, very active founding member of the Bourbaki group. His works cover many fields, his most famous contributions concerning the links between geometry and number theory.



The axiom *Ne sutor ultra crepidam* forbids us to follow him so far in his metamorphoses. [And, further on:] As for Weyl's philosophical preoccupations during this period of intense fermentation [around 1918], they soon (fortunately, we would be tempted to say) fell into a more narrowly mathematical mould. [17]

There is obviously no question of reproaching Chevalley or Weil, since they felt they were incompetent, for not technically reporting on Weyl's philosophical ideas. But the choice of words reveals a certain irony and, more radically, a profound otherness to Weyl's way of thinking and his humanist universalism. "Pushed by the milieu, no doubt, by the occasion": it sounds like the defence's plea to excuse and make up for the excesses of a young delinquent! As for the "anxiety", *it is* the characteristic of the creative genius. The technical (and structuralist) era does not bother much with metaphysical worries, but it loses in grandeur what it gains in philosophical tranquility :

[The heroism of thought] does not draw its ends and its vocation from the course of things consecrated by the peaceful and orderly system of the regime. Its justification is not in the existing order, but derives it from another source. It is the hidden Spirit, still underground, who has not yet come to a present existence, but who strikes against the present world because it considers it to be a bark which is not suitable for the core it surrounds. (Hegel [39])

Weyl is a scientist in the classical sense, that's what makes him so endearing: you can tell that he is driven by the need to know, to understand. And if that means leaving the field of mathematics, he will not fail to venture beyond it and work as a logician or philosopher.

It is true that he belongs to a transitional period in the history of mathematics and that he has been deeply marked by it. He was often able to take an exhilarating pleasure in letting himself be carried along or tossed about by the opposing currents which agitated this period, sure, moreover, deep down inside himself (as when he abandoned himself for a moment to Brouwerian intuitionism) that his good sense would guarantee him from sinking [17].

This other passage leaves us perplexed and, let's be frank, irritated. Weyl's epistemological role and ideas would thus become so many conditioned or passively undergone facts. Yet it is quite the opposite: he profoundly marked his time and, far from being drawn into it, contributed to creating the great currents of his time—as the dialogue with Hilbert testifies. As for intuitionism, it is indeed a deliberate and carefully thought-out choice on the basis of a reflection on the role and meaning of infinity in mathematics, as Weyl's numerous texts on the subject clearly attest. The fact that he was able to take Hilbertian thought into account never called into question the subordination of his activity to a requirement of radical intelligibility. In Weyl, as in Hilbert or Poincaré, thought never ceases to move, because it never ceases to adapt itself to its own process. The anxiety of their genius was not only linked to the hazards of a period of transition, but to the movements of thoughts in the making. Hence the heroism of scientific destinies which were not safe from "sinking".

This "rereading" of Weyl's work by his heirs had to be emphasised, as it is a convincing testimony to a certain mutation of mathematical thought and,

incidentally, to the way in which mathematical history may have been rewritten at one time. The leitmotif of the 1950s was the unconditional affirmation of the structural method—to which we shall return at greater length later. The great mathematicians of the previous generations were then invited, sometimes in spite of themselves, to witness the advent of the method. Their thinking was reinterpreted accordingly, embarrassing features being eventually erased or marginalized, as previously in the case of Weyl. Dieudonné, more than any other, seems suspect of having rewritten mathematical history by making it a long journey towards the domination of the structural method:

To sum up, therefore, it can be said that, for the last century, essential progress in mathematics has been marked by structural inventions. [26]

This is largely true, although the underlying concept of structure has to be agreed upon beforehand, as it has many different meanings. We will speak again later on about the ambiguities of the terminology, which allow many slips and approximations. However, even accepting a very elastic definition of structures, it is clear that Dieudonné's judgement can hardly concern Brownian motion, combinatorics, automata theory, effective and computational methods or mathematical logic. Does this mean, against the evidence, that the latter have not been marked by essential progress?

A second example will enable us to judge how what must be described as a structuralist ideology was able to impose its laws for a time, sometimes to the detriment of textual or historical evidence. It is, again, about Weyl, who published a very nice little book at the end of his life, *Symmetry*.

Starting from the somewhat vague notion of symmetry = harmony of proportions, these four lectures gradually develop first the geometrical concept of symmetry in its several forms. . . I aim at two things: on the one hand to display the great variety of applications of the principle of symmetry in the arts, in inorganic and organic nature, on the other hand to clarify step by step the philosophical significance of the idea of symmetry [89].

These lines are taken from the preface. They should undoubtedly be seen as Weyl's epistemological testament, and as an echo to the introduction of the Hilbertian *Foundations of Geometry*. Thought proceeds from intuition to concepts. Aesthetics nourishes mathematical thought which, in turn, illuminates the meanings of moments of sensitive or intellectual experience. No disjunction here between the orders of knowledge: mathematics inhabits the real world, meanings of all orders are mixed and come together to form the unity of knowledge. As for the axiomatic or structural method, it gives completion to the order of reasons. The axiomatic or structural moment is the purest moment of knowledge, but it does not dissociate itself from the previous moments. It rather legitimizes them as much as they legitimize it.

*Symmetry* proves to be one of the most moving attempts to reconcile symbolic thought with empirical insights into the world of our perceptions. It is neither a work of abstract philosophy nor a work of mathematics, as if Weyl had realised that abstract thought fails to describe phenomena or to show emotion in the face of mathematical beauty. He therefore chose to preach by example, by showing the

general public the origin and nature of hidden harmonies between the sensitive world and the world of mathematical ideas.

If the elegance of Weyl's thinking and his choice to write a hymn to beauty in his final days is confounding, the foreword to the 1964 French edition of *Symmetry* [90] is no less so, albeit in a different genre. The French title already, *Symétrie et mathématiques modernes*, suggests in itself, in its discrepancy with the English title, a reading grid and incites to mistrust. For, in 1964, modern mathematics was already no longer Weyl's, cluttered with philosophical chimeras and an outdated scientific humanism. The preface by G. Th. Guilbaud to the French edition reflects the new prejudices:

Perhaps you will understand how today's mathematicians are allowed to proclaim the decline of 'geometry'. There is no more geometry: as Descartes already knew. The object of a mathematical technique is defined only by the relations that appear in it. From the beginning of the last century (...) we realized that it was necessary to establish a rational classification of geometrical propositions (...). Driven to extreme demands, this classification, today called "structural", establishes an undisputed primacy of algebra, which is to mathematics as a whole what mathematics is to other sciences. And according to the beautiful metaphor of N. Bourbaki: "under this merciless clarity, classical geometry suddenly fades and loses its lustre". To the readers of this book I will therefore finally say: it is not the images that count here, it is what is "behind", it is their meaning—which is ALGEBRAIC<sup>8</sup> [90].

The content of this preface is exactly the opposite of Weyl's thinking, all imbued with phenomenology and the interference between the sensible and the symbolic worlds. For Weyl, and no doubt also for Hilbert, the "ruthless clarity" of the axiomatic method enhances the features of unformalised mathematics, whether geometry or algebra. Guilbaud's interpretation proceeds from the same principle as Chevalley and Weil's note: all Weyl's audacities are erased when they do not go in the direction of the new structuralist epistemology. The problem of the intuitive meaning of mathematical statements is rejected. The axiomatic method as conceived by the young Hilbert and then Weyl, i.e. the search for unconditional clarity, which for Hilbert was the model of all human knowledge, is reduced to its analytical-formal aspects alone and thus emptied of all its content.

The epistemological turning point of the 1950s is all the more surprising because it was the work of the greatest names of the time. If one chooses to oppose axiomatic method and structural method by provisionally characterizing the latter by its deliberate oblivion of logical and ontological questions, the advent of structuralism appears in retrospect, from the point of view of the fundamental problems of mathematical philosophy, more as a restoration of the ideas and problems of the beginning of the century than as a real revolution. All the questions with philosophical connotations that have been left pending are evacuated from the field of "true" mathematics, which isolates itself in the reserved domain of pure forms, of structures. The legitimacy of mathematics is thus restored, after the crisis of the foundations of the 1930s, at the price of dramatic sacrifices. Intelligibility is

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<sup>8</sup> Underlined by G. Th. Guilbaud.

sacrificed on the altar of efficiency, but mathematicians lose in the process their privileged contacts with physics and other sources of inspiration and exchange.

The structural period, after having contributed a great deal in terms of methodology, it must be stressed, has now come to an end without the following period having well-defined features. Mathematics functions well, no one can doubt it, but it would be difficult to say which theory of knowledge underlies it nowadays. Returning once again to Weyl's footsteps will enable us to sketch out the features of a thought that could replace the structuralist canons. A guiding idea underlies his intuitionist commitment and runs through his entire work, including his attempts at synthesis with the Hilbertian formalist approach: mathematics is a cultural object, it has meaning and cannot be reduced to a simple formal game. Weyl repeatedly develops the analogy between formalised mathematics and chess, to resolutely oppose the corresponding conception of mathematics. In "A half-century of mathematics", he draws the lines of the path he proposes to follow, full of nuances and conciliations. The starting point is the recognition of the axiomatic method:

One very conspicuous aspect of twentieth-century mathematics is the enormously increased role which the axiomatic approach plays. Whereas the axiomatic method was formerly used for the purpose of elucidating the foundations on which we build, it has now become a tool for concrete mathematical research [88].

The whole problem is how to interpret the successes of the method. The aim is not generality, whose possible interest is subordinated to a project and precise theoretical ambitions. The final goal is indeed simplicity and unity of knowledge. Simplicity which, according to the Aristotelian programme, lies in the search for a minimum number of presuppositions. Unity, because the same theory can be susceptible, if it is sufficiently abstract, to several competing interpretations and therefore to several fields of application.

Weyl notes in passing in the same article that, in stressing the importance of the axiomatic method, he does not want to exaggerate. The success of the axiomatic method is more relative than it seems as without inventing new constructive methods a mathematician will never get very far. Finally, he claims that it is perhaps appropriate to say that the strength of modern mathematics lies in the interaction between axiomatics and construction [88]. It is that, as a famous joke says, the axiomatic method has all the advantages of theft over honest work. Its successes are sometimes nothing more than the translation into a formalised language of results already obtained by "classical" methods and considerations. As a last resort, the great theorems always come from the study of determined objects, never from a simple set of formulas.

The successes of the formalization of algebra after Dedekind should not be interpreted as a simple victory of formal abstraction over intuition. The objects that Dedekind introduces (ideals, fields, ...) are susceptible to an axiomatic treatment, i.e. analytic-formal, but mathematicians have a perception of these objects and work with them through intuitions. These are undoubtedly more abstract than those which govern our intuition of three-dimensional space, but they exist nevertheless, and it

is they which enable us to carry out new constructions, that is to say to advance science.

As a provisional conclusion, even if it means going a little further than Weyl's message: the axiomatic method has two sides. Firstly, it is a formal-type writing process which makes it possible to codify the mathematical results in a more or less mechanical way. The corresponding writing effort has nothing creative but makes it possible to clarify a given mathematical situation. Moreover, it brings out the structuring concepts of a given problem and, as such, participates in the invention by defining new objects around which the mathematician's work is organised. To interpret the axiomatic method in the sense of formalism, that is to say by insisting exclusively on its analytical-formal aspects and by neglecting its anchorage in the phenomenality of mathematical work, is not an epistemological necessity but a choice. A debatable choice, since the meaning of the abstract concepts resulting from the method, whether they be axioms of geometry, algebraic structures or topological structures, refers to the way in which these concepts were constructed—hence to their genesis and their fields of application.

On the fringes is the more technical problem of the choice between Brouwerian intuitionism and Hilbertian formalism. Intuitionism has a precise and formalisable mathematical content but corresponds first of all to a philosophical idea which we will be interested in more than in the technical details of its constructions. Mathematical thought is based on a system of fundamental intuitions which set its domain as well as its limits. One of these limits is actual infinity, totally devoid of meaning and legitimacy. All the intuitionist choices proceed from a coherent refusal of the use of actual infinity in mathematical reasoning. This implies renouncing certain methods and results, especially in analysis, but guarantees that mathematics is a science based ultimately on what is intuitively obvious.

So far, we have not talked much about Hilbert's formalism, having focused mainly on the positions taken by the young Hilbert on axiomatics. Formalism is a radical solution, which he was only led to envisage in order to save mathematics from the intuitionist challenge and intuitionist methodological restrictions. Like many mathematicians, Hilbert refused to follow the drastic prescriptions of intuitionism, which involved painful abandonment of tools and techniques, above all in analysis. On the basis of the idea of a formal language, already put forward by Frege, formalism wants to turn mathematics into a pure play of signs. It evacuates once and for all the problem of signification by transforming it into a problem of non-contradiction of a symbolic system. Technically, the formalistic programme was to fail quite quickly, because of Gödel's famous results. However, the idea of formalism did not cease to be interesting. The position that mathematical statements would be empty of meaning must be conceived within a metatheory (the theory of demonstration, for example) as a working hypothesis rather than as an exclusive philosophical position. Moreover, an interesting point, which Weyl constantly emphasises in his confrontation with Hilbert, is that the methods of formalism relate to the problem of the meaning and legality of methods of proof. Hilbert's idea is to shift from mathematics to metamathematics, the science of logical architecture and non-contradiction, the constitutive role of evidence. In practice, the formalist

programme thus adheres in spite of itself to the founding principles of intuitionism since its methods of proof are based on finitist-type processes which lead back to a system of original evidence prior to any formalisation. According to Weyl, Hilbert and Brouwer would, in the end, agree on the essential: the finiteness of the primitive acts of knowledge, their disagreement bearing on the ground on which these acts make sense—mathematical or metamathematical.

This is not the place to develop in more detail the history and springs of the crisis of the foundations of the 1930s, nor its epistemological presuppositions. The few preceding analyses have deliberately focused on the emblematic and endearing figure of Weyl. We have deliberately chosen not to be exhaustive in our descriptions: thus, Poincaré has only been mentioned incidentally, even though he played a decisive role in the processes mentioned. In any case, when the 1950s lowered the curtain on these questions and decided to be satisfied with a Zermelo-Fraenkel-type theory of sets, they put an end to a polemic that still had a future.

A few words, by way of conclusion to this chapter, on the philosophical analog of the Hilberian journey or of the Brouwerian quest for intelligibility. As Frege's arithmetic work and the axiomatisation of geometry in the *Foundations of Geometry* had finished ruining the naïve mathematical model of mathematical knowledge, the conceptions that governed classical logic had to evolve accordingly. The desire to take into account simultaneously the genetic and axiomatic-formal aspects of mathematical knowledge was going to be translated in a philosophical context by the will to better link formal logic to transcendental logic and to delimit their spheres of legitimacy and influence. Husserl, perhaps because of his mathematical training, undertook to recast the theory of knowledge and logic in accordance with this programme. All the ambiguities and contradictions, all the hopes too, which animated the crisis in the foundations of mathematics had their philosophical counterpart. What remained for Weyl purely programmatic, because it was devoid of any theoretical justification and essentially established in the field of scientific practice, was to be the object of phenomenological analyses. The Husserlian theory of science thus provides the conceptual framework in which to formulate the fundamental problems of mathematical knowledge. It also perfectly describes the difficulties faced by thinking when it attempts to reconcile formal logic and transcendental thinking.

As the French philosopher of mathematics Cavaillès had understood in the 1940s, this conciliation itself ultimately remains as problematic and uncertain as ever. However, transcendental phenomenological idealism remains the path that contemporary thought will have to take again if it wants to reconcile mathematics and reality, whether sensitive or intelligible, one day. In *The Crisis of European Sciences and Transcendental Phenomenology* [51], Husserl observes the errors of modern science, its loss of meaning for the world of life. Of course, today the most dramatic accents of his observation seem less topical: modernity has gradually come to terms with the influence of technology in our lives and thoughts and is satisfied for the most part with a status quo that is not very glorious in epistemological terms. Thus the strictly theoretical aspects of Husserl's work have little reason to interest societies that are globally indifferent to organised thought, quickly suspected of

ideology. A question of a more existential nature, however, never ceases to arise: that of the ethical status of science, because the search for a legitimisation of formalised thought from the point of view of the “world of life” has as a correlate the possibility of giving it a cultural and humanistic function. In times of distress, the last Husserl did not cease to be concerned about it.

If we must question the meaning of science, this does not mean however that the mixing of genres that Chevalley and Weil denounce in Weyl, guilty of turning himself into a philosopher, a logician or a physicist, has to become the rule. The majority of mathematicians can continue to be satisfied with a work detached from any form of looking back at the foundations. The humanism that Husserl seeks is not sociological; it is not a question of making scientists into public persons or moral authorities: one knows only too well the dubious amalgamations and regrettable incompetence that sometimes occur when scientists believe they are authorised to intervene outside their field of competence. What is at stake is the overall attitude of a profession, its curiosity and passion for knowledge. It is also about its ability to bring its knowledge to life and to share it with others. As long as it is sufficiently publicised and disseminated, mathematics lives in the public space. This presupposes an effort of popularisation and, upstream, an adequate conception of knowledge and its anchoring in reality. Algebraic structures only make sense for a non-mathematician if they are designated as the theoretical place where a certain number of preconceptions, experiences or intuitions crystallize. Weyl’s *Symmetry* is thought in these terms, referring to an idea of science that goes beyond the technicist attitude and refuses the autonomy of pure concepts from the empirical.

But the paths traced by Husserl or Weyl are not exclusive: scientific humanism can take on other forms. Thus, according to Hilbert’s conceptions, axiomatic thought would be characteristic of theoretical thought and should therefore have an exemplary place in a project of reconciliation of knowledge. The questions of foundations are thus automatically translated into as many general epistemological questions. The idea of a formal theory of the structures of existence (formal ontology) comes from this. If we conceive formalism in terms of formal languages, the idea may seem rather vain and smoky. This is not the case if one takes the idea of pure form out of an exclusively symbolic context. We will have the opportunity to find these questions addressed in the work of René Thom for whom, it is true, they take a less theoretical turn and are more likely to be approached without resorting to a heavy conceptual apparatus.

What does this mean for thinking today? Contemporary phenomenology goes back to the problems that founded it and reads with a renewed understanding the authors whose thought influenced Husserl (Bolzano, Frege, Hilbert...). Mathematics must be part of the debate. It has everything to gain by returning to the sources of modernity and recreating the epistemological ground on which the great questions of mathematical philosophy have germinated. In this return to the origins, it can rediscover the taste for logical and philosophical questions that has been so lacking in recent decades. We have evoked some of the richest thoughts with as yet unexploited possibilities, from Hilbert to Brouwer. Poincaré, von Neumann, and many others should of course be added to them in order to obtain a faithful and

exhaustive panorama, but here we shall confine ourselves to prospective indications. All of them have seen their ideas developed in technical terms and pushed far beyond the results they themselves had obtained. This is particularly the case of Brouwerian intuitionism, which has now entered the ranks of formalized mathematics.<sup>9</sup> All these works are extremely interesting for the logician, but they have little impact on the ontological problems posed each time by this or that theory. It is therefore really on the original ground (in historical and philosophical terms) of formalism, intuitionism or any other global theory, that one should look for the ultimate springs of mathematical thought and find the corresponding meanings. It is only once this work has been accomplished that the scope of contemporary advances can be studied.

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<sup>9</sup> Heyting's notion of algebra replaces Boole's algebra in logic and allows the technical contents of intuitionism to be fully formalized.



## Chapter 6

# The Structuralist Current



**Abstract** The advent and the reign, for a moment unconditional, of the idea of structure, together with the key role played by Bourbaki in the mathematical structuralism endeavour, are central to 20th century mathematics. However, the very idea of structure is less precise than it would seem at first sight and unfolds in various ways. One is oriented towards axiomatics. Another, more programmatic, refers instead to a general regulating concept of thought.

**Keywords** Bourbaki · Modern algebra · Mathematical structuralism · Structure

During the Second International Congress of Mathematicians in Paris in 1900, Hilbert set out a list of open problems, which he rightly guessed would guide the efforts of mathematicians in the twentieth century. This speech was also an opportunity for him to affirm some general ideas that reflected the spirit in which, at the turn of the century, mathematicians conceived their science, its methods and its springs. Hilbert's presentation is resolutely exoteric and does not include any theorised epistemological views. As such, it is more consensual than Hilbert's texts on fundamental problems. His ideas have largely imposed themselves on the mathematical community, whereas the latter has only marginally responded to the interests of the same Hilbert for general questions on the origins of knowledge and the theory of knowledge. This text can legitimately be considered the profession of faith and methodological manifesto of twentieth century mathematics. We will follow its progress step by step and try to systematically identify its theoretical and practical consequences.

The first remark concerns the role of problems in the future of science. They are the ones that animate and give meaning to the work of researchers. This or that branch of mathematics is only alive if interesting problems arise. Their resolution marks the beginning of the decline of the corresponding discipline, which gradually dies out, curled up on a few questions, often technical, difficult and of limited scope. There is therefore a life of scientific corpus, which is undergoing an almost Darwinian type evolutionary process: one species of questions thrives by giving birth to new fields of investigation while another becomes extinct, the substance

that nourished it (the problems) having dried up. The particular mathematical disciplines grow or wither: first of the empirical laws of their development. When one of them is sufficiently advanced to be fully regulated, when it is frozen in a formalized text offering no new perspective, it is already dead—even if it can be reborn a few decades later if the opportunity arises. History abounds with these long-forgotten theories, which suddenly come back to life on the occasion of an unexpected discovery. The problem of defining a theory of integral calculus by finite means, which goes back to Euclid and beyond to the measurement of the volume of pyramids, is a good example. It is a question of determining to what extent can be defined on polyhedra<sup>1</sup> a theory of integration (a calculation of volumes) without transcendental methods, that is to say without argument of continuity nor passage to the limit. The problem, long forgotten after Euclid, came to light again in the beginning of the twentieth century, before sinking again into oblivion... until its algebraic springs and connections with other theories<sup>2</sup> were discovered in the 1950s.

It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon. (Hilbert, [46])

This notion of horizon is essential. It is partly found in the Kuhnian theory of paradigms, in Lakatos, although in a rather ambiguous way (among others in the idea of research programmes), and especially in Gonseth, in whose work it plays a central role.<sup>3</sup> It also has phenomenological respondents, including on the Heideggerian side of phenomenology. It is therefore a fundamental idea, often overlooked or neglected in analyses, even though it is at the heart of the meaning and motivations of scientific research. Such or such problem calls for a precise answer, but, through it, eidetic perspectives with often ill-defined contours also unfold. More than the solution of the problem, it is these perspectives which most often fascinate the mathematician, who sees in them a whole network of meanings to be elucidated, the horizon of his activity and his scientific commitment. Teaching is often reproached, and rightly so, for obscuring this aspect of scientific life by simply exposing dead and frozen knowledge. This justifies the fairly recent popularity of learning through research, which should make it possible to free oneself from a sclerotic practice of research by revealing its vitality and dynamism. The example of Fermat's theorem, recently demonstrated at the price of a long collective work, is revealing here:

The attempt to prove [the impossibility of rational solutions to Fermat's diophantine equations] offers a striking example of the inspiring effect which such a very special and apparently unimportant problem may have upon science. (Hilbert, op. cit.)

Fermat's theorem seemed too specific—too insignificant, in Hilbert's own words—to have any obvious interest in terms of eidetic content. If it continued to be the subject of hard work and exclusive passions for so long, it was because

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<sup>1</sup> Polyhedra are solids delimited by flat polygons.

<sup>2</sup> Topological groups actions, the theory of characteristic classes...

<sup>3</sup> In particular in *La Géométrie et le problème de l'espace* [35], a major work by Gonseth.

arithmeticians suspected, sometimes irrationally,<sup>4</sup> that there were hidden meanings behind this isolated result: hence the interest in a theorem whose formulation was too simple not to be part of a beautiful arithmetic theory. If the story of Fermat's theorem ended well, the importance given to a given question is often random and risky:

It is difficult and often impossible to judge the value of a problem correctly in advance; for the final award depends upon the gain which science obtains from the problem. (Hilbert, op. cit.)

The origin of the problems themselves is twofold. Firstly, there are those which come from the phenomenal world, directly or through physics. There are also those which arise in the intrinsic development of mathematics—as well as in the characterization of the properties of this or that family of objects (regular polyhedra, groups, . . .). These two sources of mathematical science—reality and the autonomous universe of concepts—come together to form the unity of a becoming and a knowing.

As for the role of axiomatic rigour, Hilbert in his 1900 speech merely insisted on its universality and its clarifying function. In his list of problems, those concerning logic—and thus the consistency of the axioms of arithmetic—are at the top of the list (second problem). The sixth problem, characteristic of the Hilbertian ambitions, is also worth mentioning: “To treat by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.”

The next point of the lecture concerns symbolism.

To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians [. . .].

The arithmetical symbols are written diagrams and the geometrical figures are graphic formulas; and no mathematician could spare these graphic formulas, any more than in calculation the insertion and removal of parentheses or the use of other analytical signs. (Hilbert, op. cit.)

Notice the pragmatism of the Hilbertian position and its insistence on the legitimacy of bringing mathematical work back to an intuitive foundation, intuition being able to take symbolic forms, which is an essential point. As for mathematical terminology, it is particularly important for practitioners, whose ideas are guided by an adequate designation of objects, be they geometric or of any other order. If a new concept is designated by a clumsy word, the imagination is quickly misled. Scientific thought has to subject itself ultimately to the rigour of form but functions usually in the ordinary eidetic mode—the one that governs all organised thought—: translating its results into formal language is done in practice at the last moment.

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<sup>4</sup> The first chapters of S. Singh's *Fermat's last Theorem* [78] show quite well the rather peculiar status of the theorem and its enigmatics components, well made to crystallize manias and passions.

This is a partial departure from Hilbert's conceptions, but formal thought is perhaps in the end only an emanation of a more original form of thought and a written demonstration, the imperfect trace (because it only takes into account the mathematical-formal aspect) of fundamental properties of understanding. This would explain why there are more or less well written mathematical texts, depending on whether they seek to account for the ideas of their authors or are satisfied with logical accuracy. To parody a formula of R. Thom: to demonstrate is not to explain. Terminological as well as stylistic choices are, within a theory, as many landmarks for the thought and help it to orient itself in the architecture of concepts and proofs.

We have seen that, in order to base mathematics on non-contradiction, i.e. on arguments of pure logic foreign to any intuition, it is necessary to resort to a finite (and therefore, as Weyl observed, intuitionist) metatheory. Similarly, it is possible to give a proof a purely logical deductive form, but the question of its meaning is then moved from the proof itself to the meaning that logical writing can have. To believe in an autonomy of the formalised mathematical discourse with regard to the meaning it has for us is an error of principle!

The agreement between geometrical and arithmetical thought is shown also in that we do not habitually follow the chain of reasoning back to the axioms in arithmetical, any more than in geometrical discussions. On the contrary we apply, especially in first attacking a problem, a rapid, unconscious, not absolutely sure combination, trusting to a certain arithmetical feeling for the behavior of the arithmetical symbols, which we could dispense with as little in arithmetic as with the geometrical imagination in geometry. (Hilbert, *op. cit.*)

These ideas have very concrete implications. In pedagogical terms, they show that it is absurd to ignore the genetic method. In France, the reform of modern mathematics and its various avatars put a lot of emphasis on discourse of the axiomatic type or, in a manner more or less equivalent at this level of teaching, of the structural type. This was justified, when the reform was undertaken, by the state of mathematics education. Today, with the help of scepticism towards the results that had then be obtained, pedagogical research questions teaching according to essentially experimental procedures—either through sociologically inspired methods of enquiry, or through the testimonies of teachers in situation. This empiricism is desirable, but often proves to be unsatisfactory. It should be supported by a general theory of mathematical intelligence which would account for its formal and genetic aspects. Establishing this theory would give a new impetus to all educational research. It could be one of the objectives of epistemology in the years to come and a project comparable in its scope to the structuralist programme of the 1950s.

But, back to Hilbert. His text closes with the testimony of his deep conviction about the unity of mathematics. He refuses the possibility of a disciplinary explosion of these, the birth of unifying concepts having to counterbalance the centrifugal and dispersive effects of scattered thematic researches. Overall, the whole text reflects the convictions of the twentieth century and “sets the tone”. Its major options continue to be shared today and continue to organise the concrete work of mathematicians. The greatest upheaval concerns the logical aspects of the Hilbertian discourse, now completely outdated—but it must be said that these aspects have

never been unanimously accepted. As for the unity of mathematics, it is highly debatable, if not on its principle, at least in the possibility of its realisation. The centrifugal movement has become too strong to be fought, and mathematics is condemned to live fragmented into more or less autonomous disciplines. If there is a conceivable unity, it cannot be a *de facto* unity, since mathematicians are now all specialists in this or that discipline (probability, combinatorics, geometry(ies), topology, analysis(es)...), but only teleologically, the hope remaining that the knowledge acquired will gradually come together, the movement of fragmentation continuing elsewhere.

After this overview of the dominant ideas in the mathematics of this century, the time has come to tackle an essential phenomenon: the advent and then the reign, for a moment unconditional, of the idea of structure. It was not restricted to mathematics and influenced fields as diverse as anthropology, psychology or sociology. In the previous chapter, the structural method could be provisionally characterized as the axiomatic method stripped of its philosophical attributes. The definition is obviously too restrictive, but was fully justified in the context in which it was formulated. For the terminology is ambiguous: it is quite legitimate to think that Hilbert, his contemporaries and his pupils could often speak of axiomatic thinking with a view to what we would today call structural thinking. Examples abound of such a usage, linked to the fact that the term “mathematical structure” only slowly imposed itself, as well as the perception of the history of mathematics that went with it. The crystallisation and fixation of the idea of structure only fully took place with Bourbaki, in whom it took on a paradigmatic status and was partially codified.

The terminological ambiguity associated with this idea has at least two facets. The first one comes from the content of the concept of axiom, which often overlaps with that of definition. Thus, today we often speak of axioms of structure about such and such a type of mathematical object. In doing so, we are only modernizing in the light of the Hilbertian works conceptual and semantic overlaps already perceptible in the Aristotelian *Organon*:

I call an immediate principle of the syllogism a thesis,<sup>5</sup> when, without being susceptible of demonstration, it is not indispensable to whoever wants to learn something; if, on the other hand, its possession is indispensable to whoever wants to learn anything, it is an axiom: there are, indeed, some truths of this kind, and it is especially to such truths that we usually give the name of axioms[...] When I [assert or deny and] say for example that a thing is or a thing is not, it is a hypothesis; otherwise it is a definition (Aristotle, [2])

In this perspective, the axiomatic way of thinking is conceived as the way of thinking that seeks the most economical and universal definitions of mathematical objects from which all their properties will be analytically deduced. It is, for the most part, the mode of thought that we qualify today as structural.

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<sup>5</sup> The thesis is what is posed without demonstration. A thesis that poses the meaning of a word but says nothing about the existence or non-existence of the named thing is a definition. If it also states the existence of the thing, it is a hypothesis. The axiom has a character of necessity and, like the hypothesis, envelops existence.

The second is related to the vagueness surrounding the idea of structure. The latter is a regulating concept of thought, designating behind such and such an object, such and such a theory, a hidden architecture. Thus we talk about social structures, the structure of a text and a piece of music or the regular structure of sea animals, starfish or sea urchins. The idea of structure thus appears, in its ordinary usage, more as a programmatic idea than as a well-defined notion. To seek to understand the structure of something is to seek its implicit determinations, and it is probably in this sense that the idea first appeared in a mathematical context. To speak of the structure of commutative groups or of the structure of regular polyhedra is to bear in mind that these objects have a certain number of characteristic properties, which could possibly be exhibited if it were necessary: in this sense, the idea of structure only partially covers the ideas resulting from axiomatisation.

Studying the structure of ellipses or commutative groups is a research programme aimed at delineating the specificities of the behaviour of certain mathematical objects which refers to the second meaning of the term “structure”. On the other hand, to speak of commutative group structure will precisely mean to have in mind the general definition of the object “commutative group”. Finally, in Bourbakist terminology, the term structure corresponds to a particular class of definitions of abstract objects, such as groups or topological spaces. It thus takes on a meaning that is significantly different from the two previous, and a more technical one.

All these distinctions may seem insignificant, but they have played a decisive role in the hermeneutic function of the idea of structure outside a mathematical context, allowing a multiform and equivocal use of the corresponding concepts. The terminology is not innocent: an axiomatic anthropology would have aroused many more reservations and less enthusiasm than a structural anthropology. Even within the mathematical literature of popularisation and structuralist “propaganda”, it is clear that the universality of the general concept of structure has favoured the success of both the term and the conception of mathematics that it conveyed. It is necessary to make here the classical differentiation between the internalist history of sciences—the history of their contents and results, a purely theoretical history and susceptible of rational reconstructions—and the externalist history, which takes into account the human data, the social and political environment and all analogous parameters. As regards external factors, it would be interesting to study the role of terminological choices in the success of scientific theories, disregarding, as far as possible, their results. In different genres, the fact that mathematical structuralism or catastrophe theory<sup>6</sup> resonated outside their own conceptual sphere and natural technical frames of reference is obviously due in large part to the analogical potentialities offered by their respective terminologies.

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<sup>6</sup> [Transl. note] Catastrophe theory is, in mathematics, a branch of geometry dealing with singularities and a branch of the theory of dynamical systems. The theory was very popular in the 1970s, as it was expected to contribute to a better understanding of biological systems. It then progressively lost momentum. We will insist later on its epistemological components which contributed significantly to the debates on mathematics, especially in France, in the 1970s and 1980s.

Returning to the beginnings of the idea of structure, still very general and with uncertain contours, we saw it germinate in algebra in the work of Dedekind. He was able to substitute a certain number of objects, familiar to mathematicians, with new concepts, in the essence of which were reflected the fundamental properties of the objects. The same method developed with Fraenkel, Frobenius, Hankel, Steinitz, Weber,<sup>7</sup> and others: the list is far from being exhaustive. Among those who were to contribute to this evolution, Hilbert appears again, this time with his work on the theory of invariants. It is also worth mentioning the work of Weyl, who gave a modern axiomatic definition of concepts such as universal covering or complex analytical variety, thus developing the axiomatic-structural approach outside the algebraic context to which its first steps are commonly attached.

In the field of algebra, the systematic shaping of the structural method, understood here as a method of substituting abstract objects (groups, ideals, rings, fields, etc.) for previously studied objects (polynomials, roots, systems of equations, etc.), is classically associated with the 1920s and with two mathematicians particularly representative of this movement: Emmy Noether<sup>8</sup> and Bartel van der Waerden.<sup>9</sup> Emmy Noether had joined Felix Klein and David Hilbert in 1915 in Göttingen, probably the largest centre of mathematical activity and creation until the beginning of the Nazi persecutions. E. Noether was soon to establish herself as the leader of the Algebraists and play a considerable role there, both through her work and her many students. She claimed the spiritual heritage of Dedekind and contributed, better than anyone else, to spreading and developing his very abstract views on algebra. In a burst of excessive modesty, she was able to say about her own discoveries: “*Es steht schon alles bei Dedekind*”, “everything is already in Dedekind”, anxious to underline the decisive influence of the latter in the formation of the methods that were finally crystallizing in the 1920s. They made it possible to reform the general idea of algebra, until then anchored in the properties of rational, algebraic, real or complex numbers. The theoretical context of Noether’s work goes far beyond that of Dedekind by situating itself from the outset in the general formalism of rings.

To use a Hilbertian terminology, mathematical thought, from the 1920s onwards, began to resolutely implement the axiomatic programme by detaching itself from objects to concern itself with their legality and abstract structures.<sup>10</sup> At the same time, the genetic point of view was gradually lost sight of, with the study of

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<sup>7</sup> A.A. Fraenkel (1891, 1965), F.G. Frobenius (1849–1917), H. Hankel (1839–1873), E. Steinitz (1871–1928), H. Weber (1842–1913).

<sup>8</sup> The most famous contribution of E. Noether is the systematic study of the ideals of a ring and the corresponding decomposition properties, a study that is resolutely in line with Dedekind’s concerns.

<sup>9</sup> Bartel van der Waerden (1903–1996), Dutch mathematician.

<sup>10</sup> Thus, topology was axiomatised by Hausdorff in 1927, the theory of operators by Stone and von Neumann in 1932, the calculation of probabilities by Kolmogorov in 1933, not to mention the attempts to axiomatise physical theories by the Hilbertian school (with Weyl and von Neumann in particular). . . There is indeed, beyond algebra, a vast movement whose heir and spokesman would be structuralism.

formal systems replacing the study of data whose existence was largely relying on evidence or intuition—such as with algebraic numbers. Noether's implicit philosophy remains however in the Hilbertian vein. It is concerned with a balance between explicit constructions and systematic analysis of concepts. E. Noether is above all convinced of the clarifying role of the axiomatic method: the accumulation of facts and concrete constructions, and the establishment of general principles that go beyond the isolation of individual facts are equally valid. The latter take factual knowledge to a new level of axiomatic understanding [1].

The real programmatic and structural turning point in algebra came with van der Waerden's *Modern Algebra*, published in 1930 [82].<sup>11</sup> Van der Waerden had been able to work with E. Noether and was aware of the major theoretical advances of the time. The German school was then at its height. The mass of results obtained, combined with the general feeling that mathematics had undergone a mutation, called for a synthesis. Among the works which appeared at that time and which aimed at exposing modern mathematics, *Moderne Algebra* was the most important, perhaps because, beyond its intrinsic technical qualities, it best reflected the mathematical sensitivity of the time. The work was immediately disseminated and became the manifesto, all over the world, of the new spirit that was beginning to reign in algebra. More than a simple factual content, it was a true conception of mathematics that it conveyed—Bourbaki and his contemporaries were not mistaken, who recognised in van der Waerden the spokesman of the structuralist movement. It was indeed with him that the idea and terminology of structures imposed itself, even if it was still devoid of any explicit content at the time and kept its most resolutely programmatic outlines.

Not to be itself axiomatised or theorised, the notion of structure acquired with him an autonomous meaning, justifying that it be made something other than a simple avatar or correlate of axiomatic thought. Van der Waerden showed *de facto* that modern algebra generated new methods and techniques—more or less uniform in spirit and independent of the particular objects studied. In other words, *Moderne Algebra* was more than just the result of a formalisation work: it corresponded to a new horizon of mathematical tools.

The case of van der Waerden will enable us to clarify the distinctions that have been made so far between axiomatic and structuralist thinking. In philosophical and foundational terms, i.e. as far as theory of knowledge is concerned, it is difficult to detect any real progress from one to the other—except that the latter focuses on a particular aspect of the former, namely the methodical analysis of concepts. On the other hand, if we consider the contents, the emergence of an original structuralist way of thinking with specific contours corresponds to the realisation that a new mathematics has been born from the process of axiomatisation, a mathematics whose strategies and methods can be conveniently designated by an adequate terminology—that of structures.

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<sup>11</sup> A detailed review of the work and review books of the German school can be found in L. Corry's [19].



It remains to be seen what the original characteristics of van der Waerden's synthetic work are. Beyond the assertion that a new, modern algebra has been born, he shows first of all that the results of the German school are organised into a theory. This theory is hierarchical, and the different algebraic objects can be distributed following an almost zoological classification, although it ultimately comes from phenomena of logical subordination. A ring is thus a commutative group with an additional composition law satisfying certain properties: its axioms of structure are richer, and the concept of ring is thus subordinated to that of commutative group. All the results obtained on the different types of objects of the new mathematics take place in an overall picture, which looks good compared to the somewhat chaotic and empirical disorder of the previous period.

But the method's achievements go further. The panorama of new algebra brings out a number of very general and extremely powerful ideas. First of all there is the concept of extension: extensions of structures on the one hand, through the hierarchy of algebraic theories, corresponding extensions at the level of objects on the other hand.<sup>12</sup> The algebraic theories are thus not only organized hierarchically: they are also nested through phenomena of transfers or extensions. To the order of logical subordination of the algebraic concepts are added phenomena of interference between these concepts. Second fundamental notion: that of morphism. A morphism is a transformation between objects of the same type which preserves the axioms of structure of the envisaged type. Or, to say it differently, a morphism is an application which preserves all the defining relations within a given algebraic theory. This concept and its different variants (injective morphisms, surjective morphisms, isomorphisms...) are today familiar to all undergraduate students. What *Moderne Algebra* establishes de facto<sup>13</sup> is their effectiveness and operative character, as well as their universality—i.e. a certain independence with regard to the type of algebraic theory envisaged.

Van der Waerden also remains very much influenced by the problems that animated algebra until the 1930s. The problems of factorization, like those considered by Dedekind and his successors, are at the heart of his treatise, but here again van der Waerden shows their ubiquity. The ideal prime numbers could be considered as the fundamental elements of the rings studied by Kummer, but in general the question deserves to be decided: what are the corresponding elements of a given algebraic structure? The answers are not always obvious, and it will be necessary to wait for the systematic development of categorical techniques, much later, to put them in some order, at least as far as terminology is concerned. However, van

<sup>12</sup> As for example when passing from the ring of integers to the field of rational numbers and, more generally, from an integral domain to its field of fractions.

<sup>13</sup> *Moderne Algebra* is, once again, the culmination of a long process of maturation. It is, for example, to H. Weber, who had an important role, that the fundamental observation that two isomorphic objects generically represent the same concept (Weber speaks of *Gattungsbegriff*) is due. For more historical and technical information on the process of constitution of modern algebra, from Galois to van der Waerden, see the article [38] (concise, but very complete) by Jean Guérindon and Jean Dieudonné, *L'Algèbre depuis 1840*, in [25].

der Waerden has the merit of indicating the problem and indicating those solutions already known.<sup>14</sup> It is worth pointing out that these questions are among the most fundamental, including for contemporary algebra and geometry. Two examples among many: the theory of group representations over finite fields or Grothendieck's conjectural theory of motives refer to them directly, but their structure is still poorly understood. To those who would therefore believe that fundamental mathematical problems have been solved long ago, such examples show that, far from it, most of the most natural questions remain unsolved<sup>15</sup>—and their definitive solution often seems out of reach in the short term.

It is, of course, doubtful whether van der Waerden and his contemporaries were aware of all the possible developments of the conception of algebra and its structures, which they were working to put in place. In any case, these developments appeared as possibilities in the horizon thus opened up for future research and were now part of the questions that mathematics would have to face. The mathematicians who will read these lines will have understood it: the algebra of Hilbert, Noether and van der Waerden went with a very open and non-dogmatic conception of the discipline, which could from then on develop in several directions. The hierarchy of algebraic theories will be studied methodically by Bourbaki, who made it the paradigm of his conception of mathematics and its architecture, but some fundamental concepts of category theory, to which we will return more explicitly later, appear in retrospect as natural abstractions of ideas from the algebra of the 1930s. This is in particular the case of all the strategies of decomposition of objects or theorems of reconstruction<sup>16</sup>—a very interesting problem, which for a long time remained rather marginal, but which has since been brought back to life.<sup>17</sup> It therefore seems reasonable to consider all these different aspects of contemporary algebra as descendants of the programme stemming from the Hilbertian axiomatic and the algebra of the beginning of the century.

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<sup>14</sup> Simple groups, irreducible polynomials...

<sup>15</sup> The concept of naturalness is undoubtedly one of the most interesting and problematic concepts in mathematical thinking. We had the opportunity to say, in the first chapter, that the theory of categories has formalised certain aspects of naturalness (those associated with transformations), but to date there is no general formalism to account for the concept (partly programmatic, such as the general concept of structure) in its full extent. In the language of mathematicians, a question is natural when it emanates with a mixture of spontaneity, obviousness and the need for a theory. If one knows how to classify surfaces topologically, it is "natural" to try to classify spaces of dimension 3, but naturalness can take more subtle forms, as in the two examples just cited (motives and group representations).

<sup>16</sup> It is a matter of reconstructing a mathematical object from partial data such as the collection of its sub-objects, the set of its representations, etc. The organisation of the dependency relations between an algebraic object and the network of its sub-objects was an important subject of study in the early twentieth century, the prototype of these results being Jordan-Hölder's theorem (1889, group theory).

<sup>17</sup> These strategies can be found in category theory with the study of Grothendieck groups, with abelian or tensor categories techniques, and so on.

In France, and for many mathematicians around the world, the idea of structure is first linked to the name Bourbaki. The Bourbaki group began to form in the years 1934–1935. Its history and the various anecdotes that accompany it are part of a certain folklore; it will suffice here to recall its main features, as well as the most significant moments. After the First World War, the situation of university mathematics in France was worrying. The republican egalitarianism had rightly ordered that privileges be, as much as possible, abolished on the battlefields and in the trenches, and mathematicians had to pay their tribute. In the 1930s, a generation was missing from the ranks of university education. The latter was based on unsatisfactory textbooks, such as E. Goursat's *Traité d'Analyse* [36]. The French university seemed to have remained outside the fundamental movement which had occurred, particularly in Germany, around the axiomatic method.

From the end of 1934, a group of young mathematicians set themselves the task of defining, “for the next 25 years”, a new syllabus for the certificate in differential and integral calculus by writing, collectively, a treatise on analysis that was as modern as possible. The project was at first very much oriented towards pedagogical concerns, but then had to evolve noticeably to move more and more towards the desire to make reference, incompatible with the requirements of a teaching manual. The list of the first members of this group, called Bourbaki, varies from one author to another according to the criteria used (assiduity, longevity, . . .). Let us quote Henri Cartan, Claude Chevalley, Jean Coulomb, Jean Delsarte, Jean Dieudonné, Charles Ehresmann, Jean Leray, Szolem Mandelbrojt, René de Possel or André Weil. It is rather difficult to identify the objectives that the group proposed to achieve, as these were to continue to evolve over the years. There is therefore a great risk of attributing to the first members of Bourbaki ideas or projects which will only be conceived or become clearer well after the 1930s. The question arises particularly when reading the articles on the history of the group: to what extent is the presentation given not a rational reconstruction, hiding processes that are much less linear and unanimous than the official history would lead one to believe?

Bourbaki cultivated secrecy, refusing to make his debates public and restricting his official positions to the publication of mathematical treatises. Contemporary historiography<sup>18</sup> has undertaken a study of the available documents, thus bringing to life the discussions and polemics that agitated the group's meetings, showing that the choices made were far from self-evident, contrary to a somewhat fixed image. Bourbaki was much criticised for his dogmatism; there is no doubt that the gradual revelation of the details of his work will do him justice and show that, behind the austerity of the facade, mathematical life was unfolding in all its richness and complexity. But all this is happening very late. The publicity of debates is essential to the life of scientific thought, and history will no doubt judge with little indulgence that the members of Bourbaki insisted on functioning for too long as a secret society, depriving the mathematical community of strategic and methodological reflections that could have been discussed in the open. A word of clarification in passing:

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<sup>18</sup> In particular L. Beaulieu, author of a thesis on Bourbaki [9].

everything we will say about Bourbaki in the following paragraphs concerns the period from its beginnings to the 1980s.<sup>19</sup>

From the point of view of the history of thought, Bourbaki played an exemplary role as a catalyst of the epistemological tendencies born of the axiomatic method and as a reorganizer of the corpus of fundamental mathematics, so that modern mathematics was often identified with the group's conception of it. However, it was not very prolific about the springs of the style it helped to create. Its only publications, mathematical treatises and their historical notes, left little room for a precise understanding of the debates that nourished them.

A small programmatic manifesto, *l'Architecture des Mathématiques* [12],<sup>20</sup> is often cited to contradict Bourbaki's epistemological silence, but it seems to have been written and published by Dieudonné, who would have presented the other members of the group with a *fait accompli* that they would have had to ratify *volens nolens*. Generally speaking, it is a whole part of Dieudonné's propaganda work concerning Bourbaki which seems questionable, since it is based on personal views attributed more or less explicitly to the group as a whole without consultation. Armand Borel, in a 1998 article [10], insists particularly and quite strongly on this and denounces in particular the abusive nature of the subtitle, *Le choix bourbachique* (Bourbaki's choice), of one of Dieudonné's most popular works [28]. All this helps to put into perspective the "ideological" part of these works, which for a long time imposed a rigid image of mathematical thought, from which it will be necessary to free oneself in order to renew its conceptual horizons.

Testimonies such as that of Armand Borel have multiplied, allowing us to see more clearly into the very nature of bourbakism. Without totally calling into question the role of spokesman that Dieudonné had for a long time attributed to himself through positions and writings which hardly admitted contradiction, the new data allow to diversify the points of view: articles of former bourbakists, such as Armand Borel or Pierre Cartier [15], autobiographies such as the *The Apprenticeship of a Mathematician* of A. Weil [85] or the *Un Mathématicien aux prises avec le siècle* by L. Schwartz [77], articles and historical studies such as those by Liliane Beaulieu. It is therefore now possible to take stock of Bourbaki's successes and failures, at a time when his work published until the 1980s can be considered part of the history of mathematics.

A page has thus been turned, and it is necessary to go beyond an idea of mathematics that has long been able to impose its programmatic and didactic views but has become positive:

To paraphrase Valéry: "We institutions know that we are mortal." If the institutional machine of the Association continues to run (very slowly), Bourbaki has fallen silent... Living mathematics is being done elsewhere, in a world shaken by the destruction of the Berlin Wall, the coming together of physics and mathematics and the invasion of computers (P. Cartier, *op. cit.*).

<sup>19</sup> This is the period during which Bourbaki published regularly.

<sup>20</sup> Published in [66].

However, this overtaking is not without its problems. Mathematicians, like all scientists, need teleological aims which enable them to structure and legitimise their discourse. Bourbaki had succeeded in giving an indisputable aura to mathematical thought, giving it a spring—structuralism—and a purpose—the search for a successful and hierarchical architecture of its concepts and results. One would look in vain today in the panorama of pure mathematics for metascientific aims that would have a similar scope, and the living mathematical philosophy, that is to say, one anchored in practice and scientific work, has been very discreet.

In any case, the time seems to have come to revisit mathematical thought and practice from top to bottom, to question his most tenacious prejudices. This book only claims to do so very partially; above all, it wants to convey a certain, current image of mathematics, in terms of its motivations, its foundations, its epistemological perspectives. The way it would recommend, if a new theory of mathematical knowledge had to be elaborated, is one of the most classical: to relearn to make the part in mathematical objects of the form and the meanings. That is, developing a very widespread idea, but which still lacks a support and a justification in terms of general theory of knowledge:

Most books nowadays tend to be too formal most of the time, they give too much in the way of formal proofs, and not nearly enough in the way of motivation and ideas. Of course it is difficult to do that—to give motivation and ideas. There are some exceptions. I think the Russians are an exception. I think the Russian tradition in mathematics has been less formalized and structured than the Western tradition, which is under the influence of French mathematics. French mathematics has been dominant and has led to a very formal school. I think it is very unfortunate that most books tend to be written in this overly abstract way and don't try to communicate understanding. (M. Atiyah,<sup>21</sup> [7])

This means, incidentally, recognising the interest and legitimacy of fundamental questions, even when they are formulated in the most radical philosophical or logical way (formal languages, syntax, phenomenology...). These are all ways that Bourbaki rejected as not being within the domain of mathematics, by imposing a conception deliberately stripped of all metascientific aims. However, this choice was made in the name of arguments and opinions which are beyond the competence of mathematics, and which emerge from a perception of science whose only justification is pragmatism and a certain interpretation of the formalist programme. This is not enough to guarantee its intellectual legitimacy, all the more so as the limits of formalism were very quickly apparent to Hilbert, Russell and other theorists of mathematical thought. From now on, we are going to discuss these aspects of Bourbakism, while continuing to leave aside all that relates to the content of its contributions, remarkable and indisputable, in the mathematical field itself, that is to say all that relates to the content of its treatises and the shaping of the corpus of fundamental mathematical concepts that goes with them.

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<sup>21</sup> M. Atiyah was awarded the Fields Medal in 1966 for his work on K-theory and index theorems, very abstract and algebraic subjects: it is as a practitioner of very theoretical mathematics that Atiyah questions the formalising approach, its excesses and the need to re-emphasise the meaning of mathematical thinking.

Bourbaki's early work is strongly influenced by van der Waerden's style of structural mathematical writing and, beyond that, by the Hilbertian tradition. The originality of the Bourbakist endeavour lies first and foremost in its universal vocation. The first project, to write a treatise on modern analysis, had expanded quickly enough to cover most areas of pure mathematics. Making axiomatic rigour, which had become the rule in algebra, the norm for all mathematical writing, was undoubtedly the first Bourbakist achievement. It was around it that the first oppositions to a style described as very dry, too rigid in its organisation and difficult to read, among other things because of a system of cross-references that was rather cumbersome to handle, also crystallised. But, more than the style, it was a certain evolution which was denounced: evolution towards generality and abstraction for their own sake, without references to the fundamental problems which make the life and soul of mathematics. Bourbaki always strongly defended himself against these accusations, sticking to the Hilbertian line: abstraction is justified if it brings new tools and a better understanding of fundamental phenomena. Was it for all that justified to systematically prevent any concession to the genetic method?<sup>22</sup> In retrospect, it is legitimate to doubt it, because it is to ignore the meaning of the results, but it is it which guides the instinct of the mathematician and makes his passions live, and not the fixed and lifeless structures of formalized mathematics.

Thus the refusal to make any stylistic concessions seems in retrospect to run counter to one of the fundamental Bourbakist requirements, which was to provide mathematicians with working tools: great works are rather nourished by great programmatic ideas than by precise and detailed results. Several former bourbakists, foremost among them Chevalley, have retrospectively recognised that formalised and completed mathematics was a "dead mathematics", no longer brought to life by the breath of thought. Once crystallised, imprisoned in a formal language, ideas lose their content and become tautological. It is a frequent feeling, and common to all researchers: the feeling of emptiness, of a somewhat vain obviousness, that a barely obtained theorem can give. As if months or even years of work led to a result which loses all its charm when it is discovered and codified according to the mechanical rules of axiomatic writing. The stylistic choices contribute to such a sensation, and there is no doubt that bourbakist writing lends itself particularly well to such a disenchantment with the world of constituted mathematics. The reader of Bourbaki must then take it upon himself to revive the original intuitions underlying the sanitised writing of the treatises, appealing to his imagination or his culture to do violence to the signifying neutrality of the texts. To tell the truth, Bourbakist treatises are excellent as long as one knows their content in advance. They then perfectly fulfil their role as a "toolbox".

That said, more than the treatises themselves, the "Bourbaki spirit" was to have a profound influence on the 1950s.

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<sup>22</sup> As opposed to the axiomatic one: the idea of the genetic method in teaching is to build progressively concepts and theories, following for example their historical development or successive advances in the solution of problems.

The fifties was a period of spreading influence of Bourbaki, both by the treatise and the research of members. Remember in particular the so-called French explosion in algebraic topology, the coherent sheaves in analytic geometry, then in algebraic geometry over the complex numbers, later in the abstract case, and homological algebra. . . .

The influence of Bourbaki on one's work and vision of mathematics was obvious to many in my generation. For us H. Cartan was the most striking illustration, almost an incarnation, of Bourbaki. He was amazingly productive, in spite of having many administrative and teaching duties at the École Normale Supérieure. All his work did not seem to involve brand new, groundbreaking ideas. Rather, in a true Bourbaki approach, it consisted of a succession of natural lemmas, and all of a sudden the big theorems followed. (Borel, [10])

The methodology described by Borel thus permeates and influences the research community and goes beyond the strict framework of Bourbaki to be the fact of an entire era. The idea behind it is resolutely in the axiomatic tradition: one must always go to the essential, to the most fruitful, the most general, the most fundamental concepts. Here we find again the teleological idea of structures, such as it was already present in van der Waerden, but the power of the idea has moved from the algebraic field alone to all mathematics. The whole exegetical difficulty lies in the dichotomy that arises between this aspect of bourbakism—a certain way of doing mathematical research which is done by the members of the group as autonomous scientific individualities—and the discourse of Bourbaki as such, that is to say as the author of a mathematics textbook. To use a political analogy, the general will must not be identified, in a Republic, with the mere sum of individual wills. In the same way, the intellectual and moral personality of Bourbaki is to be distinguished from the sum of the personalities which compose him. This dichotomy has often been stressed, and commentators have not failed to point out the obvious discrepancy between Bourbaki's official discourse and some of his members' practices, that did not quite conform to the orthodoxy of the group. This schizophrenia is evident in the use of vocabulary and categorical techniques. It could be at the origin of the progressive decline of the Bourbakist writing enterprise which, in order to be able to revive effectively from now on, would have to be radically renewed in its aims and methods.

In the next chapter, we shall return in greater detail to the nature of Bourbakist epistemology and metamathematics, concluding the present one with a brief analysis of the evolution of his projects from the 1930s to the 1980s, following essentially the description given by A. Borel. Broadly speaking, the history of Bourbaki can be divided into two periods: from the beginnings to 1958, from 1958 to the 1980s. The first period concludes with a 1958 talk by H. Cartan, "Nicolas Bourbaki and contemporary mathematics" [14]. At this time, Bourbaki has more or less completed his initial programme: writing a treatise on the foundations of Modern Analysis. The treatise then consisted of eight books on Set Theory, Algebra, General Topology, Functions of a Real Variable, Topological Vector Spaces and Integration. Bourbaki then faced an internal crisis. The hope of writing a text totalling all fundamental mathematical knowledge became illusory—it would have required much more extensive human resources and a change in methodology. Let us listen to Cartan:



It cannot be denied that the Bourbaki of today has new difficulties to overcome. Mathematics has changed radically in the last 20 years (a development in which Bourbaki himself may have had a hand). There may be some concepts among the fundamentals in Bourbaki's textbook which have already become outdated. After Bourbaki finishes Part One—"Fundamental Structures of Analysis"—he may feel obligated to start all over again. But Bourbaki will not restrict himself to the fundamentals of mathematics. What are his plans now? That I may not divulge. But one thing is certain: if the goals change, so must the methods. The final result remains shrouded in the future. Perhaps it can be discussed in another 20 years. [14]

As the members of Bourbaki are supposed to retire from the group at the age of 50, a new generation takes over. Let us mention Pierre Cartier, Samuel Eilenberg, Alexandre Grothendieck, Pierre Samuel or Jean-Pierre Serre. The idea of a linear ordering of the treatises, of a perfect hierarchy of their contents, is no longer tenable. Moreover, the axiomatic style is now sufficiently widespread for textbooks to be published, independently of Bourbaki, which for the most part satisfy Bourbakist methodological requirements, thus making it unnecessary for Bourbaki himself to write treatises on the same topics. The work of Oscar Zariski (1899–1986) in algebraic geometry thus offers an example of a success of the axiomatic-structural method inherited from the German algebraic school that was obtained independently of the Bourbakist enterprise. Zariski was trained at the Italian geometric school (Enriques, Castelnuovo...). Unhappy with the methods used, he began as early as 1937 to reform algebraic geometry in the spirit of modern algebra. The geometric school founded by Zariski (Artin, Hironaka, Mumford) achieved major results and profoundly influenced modern algebraic geometry. The members of Bourbaki also published various treatises (on algebraic topology, homological algebra, etc.), under their own name and on an individual basis. If the method is thus uniformly recognised and accepted, a sign of undeniable success, the originality and specificity of the Bourbakist endeavour and the work carried out in its name are all the more reduced.

Finally, another of the group's postulates is undermined: the rule was that each member of the group should aspire to universality and be interested in the entire mathematical corpus, according to a tradition dating back to Poincaré. Some manage to do so with astonishing ease, like P. Cartier, but the rule can hardly continue to prevail uniformly because the extent of mathematical knowledge is constantly exploding and the threat of specialisation lurks for the members of the group. Two main directions are emerging. To continue, in the tradition of the first Bourbaki, to build a foundation for future mathematics, going beyond the elementary contents. Or, in a more pragmatic way, to engage in the writing of more specialised textbooks, abandoning the "Elements of Mathematics" approach, i.e. breaking with the Euclidean vein of the early days.

According to Armand Borel's testimony, Alexandre Grothendieck appears at this moment of the Bourbakist journey as the "heroic figure" of the group: the man able to regenerate the initial project and give it a new impetus. Grothendieck's entire work bears the mark of the greatest aspiration of bourbakism: the search for the right concept, in the optimal degree of generality. The path he opened up represented a



new qualitative leap for the axiomatic programme of the Hilbertian tradition. Within the framework of Bourbaki, Grothendieck would propose to define and structure the regulatory concepts of algebraic topology and differential, analytical and algebraic geometry.<sup>23</sup> If his programme had been adopted, the Bourbakist effort would have changed register by tackling the ordering of contemporary mathematics. This was not the case, but, in Grothendieck's titanic work, we can see the effort of a man—a hero in the Hegelian sense, fulfilling the destiny of an idea—to carry out a radical upheaval of the fundamental ideas of geometry and topology.

The immensity of the work to be undertaken, the uncertainty of success, a certain mistrust of Grothendieck's immoderate taste for endeavours at the extreme limits of abstraction, were going to lead Bourbaki down other, more modest and more reasonable paths. Three books came out of it: *Commutative Algebra*, *Lie Groups and Lie Algebras* and *Spectral Theories*. *Lie Groups and Lie Algebras* was a success, showing that the choice of specific themes could lead to unquestionable results.

Yet it emerges from Borel's testimony, on which our analyses are based, that Bourbaki seems to have suffered from a lack of overall purpose. The edifice of mathematics is not constructed mechanically: it needs, in order to order itself, regulating concepts which go beyond the technical horizon. The search for a field of original concepts for topology and geometry could have played this role, but then Bourbaki would probably have had to question the vision that his founders had put forward of mathematical structuralism.

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<sup>23</sup> Sheaves, categories, homology, coverings....

## Chapter 7

# Structures and Categories



**Abstract** Bourbaki would have finally stumbled on epistemological difficulties: his initial programmatic conceptions proved unsatisfactory in view of developments in contemporary mathematics, which would require new ideas and new ambitions. This chapter looks further into Bourbaki's mathematical epistemology and investigates its connexions with category theory.

**Keywords** Bourbaki · Category · Lawvere · Structure

Bourbaki would therefore have finally stumbled on epistemological difficulties: his initial programmatic conceptions proved unsatisfactory in view of developments in contemporary mathematics, which would require new ideas and new ambitions. In this chapter, we will look further into Bourbaki's mathematical philosophy, which is, with historical hindsight, fairly easy to reconstruct, despite the few direct testimonies available. Everything points to the content of Hilbert's speech of 1900, whose guiding ideas run through the texts of Bourbaki and those of his collaborators who spoke out on methodological issues. More generally, Pierre Cartier underlined Bourbaki's dependence on the most traditional currents of German thought—a certain form of rigorist idealism:

The Bourbaki were puritans, and the puritans are strongly opposed to pictorial representations of the truths of their faith... Bourbaki is the spiritual son of German philosophy. He was founded to develop and propagate German philosophical views on science. André Weil always liked German science, and he always quoted Gauss. All these people, each with their own personal tastes and opinions, were proponents of German philosophy. And there was this idea that there is an opposition between art and science. Art is fragile and deadly, because it appeals to emotions, visual meaning, and informed analogies. [15]

To this should be added a clear, albeit ambiguous and somewhat reluctant, dependence on Russell. Many things therefore tend to dissociate Bourbakist thought from a certain mode of thinking “à la française” and, even if Bourbaki ended up claiming a filiation with Poincaré, it is anything but obvious. He came closer to Poincaré only after long debates, and in any case the opposition in style is obvious between them. There is, however, a “French” feature of Bourbakism: Jacobinism, but it is more political than philosophical.

In any case, the most direct and readily acknowledged influence is that of Hilbert, who is both manifest and indisputable, even if Bourbaki retained above all his exoteric views on science. In the first place, Bourbaki conceives the axiomatic method as a quest for additional unity and clarity. The systematic search for an adequate terminology or the taste for rigour are also presented as founding principles of his writing, rules which have since been imposed on the entire mathematical community. In detail, a number of specific and more original features give his thought its own tone. The organisation of knowledge is one of his dominant preoccupations. At the van der Waerden school, Bourbaki learned that mathematical concepts are arranged in a hierarchy, but he theorised and systematically extended to all mathematics what had remained implicit in *Moderne Algebra*.

That mathematics has a conceptual ordering has been clear since Aristotle, who already knew that science can be organised, with the axiomatic method providing the rules of the game. That this ordering has a describable legality, i.e. an intrinsic architecture other than that resulting from the laws of logical dependence, is, on the other hand, not at all self-evident. The axiomatic method learns to look for the primary reasons (axioms), but says nothing about their organisation. Logic says nothing more, since it teaches on the contrary the neutrality of meanings and allows only obvious distinctions (axioms stronger than others,...). Bourbaki, and this is his specificity, claims to impose on mathematics a hierarchical and rather rigid organisation with, upstream, an axiomatic theory of sets (Zermelo-Fraenkel) and, downstream, a “classification of fundamental disciplines” which justifies the deductive structure of the treatises. In the latter, the principle emerges of a systematic indifference to the applications of mathematics or to the disciplines which do not blend into the chosen system. Mathematical physics, combinatorics or probabilities have had to suffer from these discriminations, which appear quite surprising with hindsight.

Of course, Bourbaki had to give a conceptual justification for this “architectural” wager: pragmatism alone could not suffice to underpin a normative theory of mathematical content. As Bourbaki has always kept away from logical and philosophical debates, preferring to stick to the opinions of the “mathematician at work”, he had to find a justification for his methods that did not resort to traditional metamathematical or epistemological tools. The idea emerged that it should be possible to account for the conceptual organisation of the mathematical corpus in strictly mathematical terms. This is what had just happened, shortly before the constitution of the group, with modern algebra, the emergence of unifying and structuring concepts having made it possible to reshape the discipline. But, in algebra, the phenomenon had occurred of its own accord, as a natural consequence of the progress made until the early 1930s, and it had not been theorised: Van der Waerden merely described the logical dependence of the concepts; the theme of algebraic structures retained uncertain contours and all its programmatic generality.

More ambitious, Bourbaki wanted to take mathematical thinking to a fundamental turning point, not unlike certain Leibnizian impulses, by giving the concept of structure a precise mathematical content. In other words, he sought to axiomatise and codify the very edifice of mathematics through a systematic theory. This is the

heart of Bourbakist thought, the central element that justified most of his strategic positions, but also the most debatable point of his entire work.

Bourbakist theory distinguishes three orders of fundamental notions: algebraic, ordered and topological. As he methodically refuses to use epistemological arguments other than pragmatic or strictly technical ones, the reasons for the choice only make sense in the light of the history and initial ambitions of the group (writing a treatise of analysis). Algebraic objects (groups, fields,...), ordered sets and topological spaces present a common characteristic: they can be described in purely set theoretical terms with the help of a few elementary notions (Cartesian products, power sets...). Ad hoc conditions on this type of constructions make it possible to affirm that we are dealing with a given type of algebraic object, an ordered set or a topological space. Hence the idea of defining a universal notion of “structure”, which would encompass all these situations and, more generally, would contain in power all the possible definitions of mathematical objects through an adequate use of fundamental operations on sets. In seeking in this way an unconditional generality, Bourbaki came up with a very precise concept of “structure”, but difficult to use—apart from elementary situations where one could very well dispense with using it. This is evidenced by the fact that Bourbaki himself made little use in his treatises, and only anecdotally, of his own formalized notion of structure.

Last point: the chapter on structures in *Set Theory* only appeared in 1957. Mathematics has then evolved a lot since the birth of the group. At that time, without even mentioning its mathematical efficiency, the hermeneutic scope of the formalised concept of structure became debatable. New concepts, essentially linked to the theory of categories, were to compete with it for the privilege of claiming to organise, at least partially, mathematics. History is advancing at a rapid pace: what was progress in the 1930s became fixed in positive knowledge at the end of the 1950s. Bourbaki’s refusal to take these changes into account and the increasingly obvious inadequacy of his implicit mathematical philosophy have always surprised commentators, all the more so since the paradigmatic notion of structure immediately runs counter to one of the group’s methodological postulates: never introduce generalisations deprived of effectiveness, systematically refuse abstraction for its own sake:

An attack on Bourbaki’s point of view, [claims] that he would have encouraged the tendency to publish empty mathematics and to do generalizations for generality’s sake. Whenever it was decided to include a notion or a main theorem in Bourbaki, it was only after prolonged discussion, in which its advocates were required to show in which results these notions or results were a crucial tool. (Dieudonné, [27])

The introduction of the concept of structure, in the form it takes in the treatises, does not meet these criteria, even though it intervenes constantly in the Bourbakist epistemological system.

A testimony by S. Mandelbrojt expresses, in this respect, in a very elegant way the subtle balance in the search for generality that mathematicians pursue. It is an individual testimony, but which expresses a view shared by all the supporters of the

axiomatic method, who (like Bourbaki) refuse to adhere to a strict formalist point of view:

Being general is a great virtue for a mathematical fact, and one feels a kind of slight contempt for the particular fact.

However, I can't divinise the generality in itself... Generality is beautiful when it has an explanatory character. When one discovers a theorem with restrictive hypotheses which one feels little linked to the subject or which seem to be due only to the method used for its demonstration, one tries, obviously, to get rid of these awkward conditions. So we try to generalize the result to give the phenomenon its natural aspect... Personally, I feel that there is an optimum to this generality (S. Mandelbrojt, [71]).

The regular practice of rigour and formalism has taught mathematicians the easy use of abstraction: the search for generality in formulation is no longer the heroic act performed by a Dedekind or a Hilbert. The scope of abstraction is reduced all the more; from now on, it must be justified by leading to the implementation of new fields of operations so as not to be reduced to a simple writing game. The notion of balance is also essential: the right mathematical writing must express a relation of convenience between things—between forms and contents, symbolism and meanings.

Abstraction is beautiful and great when it is explanatory, when it gives, I dare say, a metaphysical character to the phenomenon. In a word, when this abstraction indicates the world which is proper to the discovered phenomenon. But if one loses the concern of searching for this world which is proper to it and which one seeks to generalize only by a taste for generalization or a taste for abstraction, one risks entering a formalized world. Now I like, in a mathematical theory, the presence of matter. (S. Mandelbrojt. op. cit.)

Although it is largely devoid of a convincing explanatory character, the Bourbakist theory of structures nevertheless has a strong metaphysical character which no doubt explains the unconditional and rather unfortunate attachment to it. From the point of view of its finality, it has the attributes of an *Arithmetica universalis* or a *Characteristica universalis*. It is supposed to provide the foundation on which to build deductively living mathematics, with the greatest possible degree of eidetic clarity—a bit like the Hilbertian axiomatic is the perfect ground on which to reconstruct all Euclidean geometry. The ambition was beautiful, perhaps it would have required more audacity and less dogmatic rigidity in its accomplishment!

The precise reasons for the failure are difficult to understand in retrospect, especially since the competing theories to the Bourbakist theory of structures<sup>1</sup> were squarely into the “structural” thematic horizon and can appear as examples of research that the group could have systematically supported.<sup>2</sup>

<sup>1</sup> Universal algebra or category theory. One can also quote here the lesser known combinatorial work of A. Joyal on structure species or Lazard's theory of analyzers.

<sup>2</sup> With the notable exception of Ehresmann, who was one of the (temporary) members of Bourbaki most attracted to the “metaphysics” of fundamental concepts. But, although considerable, Ehresmann's work remained rather marginal in French mathematics, and he himself seems to have experienced a painful isolation due to his thematic choices.

In fact, if the idea of a bourbakist structure has no real operative function, it is first of all because it has remained in an embryonic state and does not seem to have ever been considered as an active field of research by the members of the group. In retrospect, the feeling dominates that its members were able to identify structural problems as one of the major challenges for the theory of knowledge, but without ever really taking it seriously as such. All of them were probably too busy with their individual successes in geometry and related disciplines to spend too much time exploring questions of eidetic foundations, even if they were posed in technical terms. Perhaps the examination of Bourbaki's archives will shed some light on this astonishing missed opportunity.

On the fringes of the theory of structures, the Bourbakist ideas on foundations have been violently criticised by a logician, A. Mathias [72]. His attack partly echoes the reservations that can be expressed about the idea of structure: Bourbaki would never have really taken logic or epistemology seriously. Mathias denounces the approximations concerning the Zermelo-Fraenkel system or a lasting misunderstanding of Gödel's results. In its most extreme form, Bourbaki's attitude on these questions is characterised by a famous description by Dieudonné:

Concerning Bourbaki's attitude towards the problem of "foundations": it can best be described as total indifference. What Bourbaki considers as important is *communication* between mathematicians; personal philosophical conceptions are irrelevant for him. [27]

Today we must put an end to such principled positions. If the failure of the structuralist programme translates into the necessity to give back to the concept of structure its teleological function without trying to freeze it in a univocal formalized mathematical notion, Gödel's work shows, as far as the foundations are concerned, the limits of the strategy consisting in being satisfied with a Zermelo-Fraenkel-type system and disinterested in metamathematics. We include here in metamathematics synthetic and prospective views, such as the search for concepts originating a discipline, a search which is not the direct responsibility of formalised mathematics. In other words, in order to go beyond Bourbaki today, it is necessary to put an end to pragmatic discourse and to restore, alongside research, philosophical debate. Mathematics has everything to gain from this: it is the only way for it to win back an audience. The media success of physics, its immediate rival in the pantheon of pure sciences, is due to the fact that its most fundamental questions have managed to strike the collective imagination—just think of the interpretation of Copenhagen!

The need to revive tomorrow an ambition as generous as the one that animated the young Bourbaki seems obvious:

I believe it is a pity that the sense of unity and intellectual excitement in mathematics my generation experienced in its youth, which to a great extent we owe to Bourbaki, has dissipated into a sort of creative anarchy [...] At some point we will have to reach again for the grand generalization and unification. (R. Hermann, [44])

It is difficult to specify its features, as this would be anticipating the evolution of mathematics. Two facts are inescapable, taking into account the state of mind of contemporary works. First of all, the individual reaffirms its rights. After a period of creation of great theories, modernity is very much concerned with the

effectiveness of general concepts. Where such general theory predicts the existence of characteristic phenomena, these often remains to be described. The rise of computer science is part of this movement, but the computational trend affects all sectors, including those where calculation remains essentially virtual and abstract (as in algebraic topology). Like the desire to generalise, the will to calculate is however only subordinate to a project, to the search for an additional meaning, and as such is a matter of strategic choices. Taking into account the different aspects of effectiveness and computability will be one of the duties of a unified and renewed theory of mathematical knowledge.

Another of its characteristics will concern the conceptual contents. As the achievements of axiomatic rigour are uniformly accepted, everyone now also agrees on the need to add “matter” to formal notions, that is to say, to concern oneself in any mathematical text with intelligibility as well as coherence.<sup>3</sup> Make no mistake, this is the most difficult exercise. Translating the idea of a demonstration into formalized language is a simple matter of patience, as long as the idea is correct. Describing this idea, explaining a motivation, is otherwise difficult, because it is a matter of style and imagination. Experience teaches that the game is dangerous, and its rules remain to be codified—but the Erlangen programme or the Grothendieckian “oral tradition” are there to show that the greatest mathematical ideas are not necessarily conveyed by formalised texts. The latter are all too often the unsatisfactory residues of a great intuition, betrayed or amputated by standardised writing and its necessary simplifications.

It will also have to account for the logic of the transfer of intuitions. In its progression, mathematical knowledge produces increasingly abstract concepts, which nevertheless keep as the imprint of original intuitions (spatial, arithmetic, combinatorial,...). Thus the algebraic topologist is often dealing with “spaces” which no longer have anything spatial in the current sense of the term, but which can still be referred to geometric intuitions—such as the ability to be “cut out” or “glued” in pairs. The practising mathematician knows how to use these intuitions outside their context of naive evidence; he also knows their limits when applied to abstract objects. This skill is often the result of long work. For mathematics to live, it must be easily communicated to novices so that they can one day take over and go further than their predecessors. In other words, mathematical knowledge is also for many a know-how, the rules of which are those of a technique as much as those of formal knowledge. Textbooks designed according to the rules of the structuralist method often leave a feeling of incompleteness: the reader has understood the springs of the method, but would be incapable of making it work in the study of concrete situations. He who is content to transmit his results to the general public, without communicating to it the capital of intuitions and eidetic points of reference which goes with it, has only done half the work, just as a craftsman who would keep his manufacturing secrets to himself could only have poor apprentices. Communicating implicit strategies is the hallmark of a true mathematical education.

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<sup>3</sup> This is the meaning of the quote from Atiyah in the previous chapter.

It is also the condition so that science does not become sclerosed or fragmented into neo-pythagorean schools, jealous of their specificities and their secrets.

If it refused to consider the unsaid of knowledge, one of the most remarkable features of Bourbakism is to have understood that science is a collective work and, even more radically, anonymous—for the concepts are autonomous and their paternity illusory.

Whatever was accepted would be incorporated without any credit to the author. Altogether, a truly unselfish, anonymous, demanding work by people striving to give the best possible exposition of basic mathematics, moved by their belief in its unity and ultimate simplicity. (A. Borel, [10])

This human component is far from negligible, and the few methodological criticisms that can be levelled at Bourbaki in retrospect should not conceal a great intellectual generosity—a generosity that is surprising today, when the publish or perish rule and competition have taken hold just about everywhere. The spirit and morals of the 1930s were different. The atmosphere surrounding his birth brings to mind the circle of young creators in Aragon’s novel *Aurélien*—the expression of a creative strength and carefree spirit, against the backdrop of the butchery of the first war and the tacit will to change the order of the world. Bourbaki realised, at the time when he was writing his treatises, with the Borel, Cartan, Cartier, Chevalley, Dieudonné, Weil, to name but a few, an ideal of humanist and disinterested collaboration in the progress of science, thus giving figure and body to the Husserlian image of the civil servant of humanity. Let us also mention Borel and the conclusion of his testimony:

What remains most vividly in my mind is the unselfish collaboration over many years of mathematicians with diverse personalities toward a common goal, a truly unique experience, maybe a unique occurrence in the history of mathematics. The underlying commitment and obligations were assumed as a matter of course, not even talked about, a fact which seems to me more and more astonishing, almost unreal, as these events recede into the past. [10]

It is difficult to say what bitterness can be involved in Borel’s remarks, which are very harsh towards a modernity where this ideal has taken on unreal contours! However, let us mention that C. Chevalley, in an interview, “Nicolas Bourbaki, Collective Mathematician” [18], gives a retrospective judgement much more nuanced and critical than that of Borel. He also acknowledges the idealism of the group’s beginnings, but very explicitly questions its evolution. It would have become little by little a place of academic and symbolic power, thanks to its organization in a “secret society” where members were co-opted and influential (or called to become so). Sociology such as Bourdieu’s, if it one day looks at power networks in mathematics, will no doubt be able to tell us more.

Let’s get back to mathematics. If the success of the theory of structures has been more than uncertain, another theory has at the same time partially imposed itself in its place: that of categories. Contrary to the first one, the latter was born independently of any architectural aims. It is a purely mathematical theory, even if it has sometimes come to claim a foundational role and compete with set theory. It was born in the mid 1940s. The first articles on the subject had an essentially semantic



function, they gave a name to more or less classical procedures and objects, but their mathematical content was not really convincing at the time. One of the characteristic features of the history of the theory manifests itself here: before providing specific working tools, it is first of all a convenient and efficient language of description, a universal language in the sense that it concerns most of the fields of activity of pure mathematics. Even if it were to be reduced to this simple linguistic function, the theory would be worth adopting because it responds to a concern for conceptual and terminological unity.

In a nutshell, it provides the right framework to talk about transformations, whether they are internal transformations within a theory (algebraic, topological, etc.) or the transition from one theory to another. Bourbaki adopted a very ambiguous attitude towards it, not knowing how to integrate it into his architectural conception of modern mathematics (while stumbling over problems of axiomatisation of the theory). It is reasonable to suppose that by showing *de facto* that one of the key ideas of algebra (the behaviour and intrinsic combinatorics of morphisms<sup>4</sup>) was just as fundamental as the notion of structure as it had been formalised by Bourbaki, categories ruined the image of mathematics that Bourbaki proposed to privilege in his treatises.

From the beginning of the 1950s, particularly with the work of Eilenberg and Steenrod and their categorical axiomatisation of topology [30], the technical effectiveness of categorical concepts was established: they showed that, far from being reduced to mere language, categorical thinking allows a better understanding of mathematical reality and, above all, allows new results to be obtained. These results, without necessarily being very complex, prove to be fundamental from the outset. It becomes clear that categorical language makes it possible to formalise a whole set of intuitions, essential in algebra or topology, which had remained implicit for too long due to the lack of adequate concepts with which to express them. From this moment on, the theory is no longer a simple mode of expression which can be circumvented. It imposes itself as one of the basic tools of contemporary mathematics and, in an equally essential way, as the receptacle of original and irreducible ideas.

The notion of transformation or, in a more orthodox terminology, of morphism (when it is a question of transformations within a theory) and of functor (when it is a question of transformations between theories), generates a logic of its own and follows combinatorial and behavioural rules whose meanings go beyond the framework of Bourbakist structures. To put it another way, axiomatizing the idea of transformation is a programme just as legitimate and primitive, from the point of view of general mathematical knowledge, as axiomatizing the typology of mathematical objects.<sup>5</sup> The moral of these considerations is that there is no primary intra-mathematical theory on which to base mathematics. Whether it is a

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<sup>4</sup> That is to say, independently of the objects themselves and their individual structures.

<sup>5</sup> Church's Lambda Calculus, which plays a role as a foundation theory for theoretical computer science, also goes in this direction.

question of sets, functions, structures (Bourbakists), categories, or other objects, such as topoi, which also have a foundational vocation, assigning them each time a privileged and exclusive role is equivalent to favouring a particular type of intuition, a specific mode of functioning of mathematical intentionality. This can only be done by dramatically amputating the field of thought, or by artificially distorting its organisation. Even if the architectural (such as Bourbaki's) or foundation theories (such as set theory) which have been proposed so far would be fully satisfactory from a technical and logical point of view, they would nevertheless remain flawed by choosing to exacerbate a specific aspect of thought.

It would, of course, be pointless to ignore the technical aspects of mathematical research in such considerations. It is they who have, to a large extent, contributed to promoting a certain type of logical and axiomatic totalitarianism. However, history seems to establish the primacy of the conceptual (and aesthetic) part of the edifice of mathematics over the technical component which, like stewardship, must follow but does not have to impose its views: it is more fruitful to draw up an evolutionary table of the fundamental concepts of the corpus already constituted, and to proceed from there to build a synthetic and unifying work, than to pretend to deduce everything from the more or less forced axiomatisation of an isolated concept.

For this reason, it is moreover legitimate to doubt the validity of attempts to make categories an exemplary primitive concept, for example by seeking to erect the category of categories as an organisational concept, thus reproducing with new concepts the error that had been made with structures. The theory of categories can be formalised in an adequate version of set theory, which means in concrete terms that the passage from one theory to the other does not alter the content of the fundamental difficulties. The interest of this type of research is therefore first of all technical, it allows us to refine our understanding of systems of fundamental axioms, but it is doubtful that through them more than a simple enrichment of the technology can be achieved.

Rather than by its fundamental aspects, category theory is therefore worth, from an epistemological point of view, by the concepts it introduces into the field of constituted mathematics. We have already discussed the emphasis placed on the notion of transformation and, more generally, the upheaval that this has induced in the hierarchy of fundamental concepts. In a more specific way, some categorical phenomena seem to have a real philosophical content. Of course, one must be wary of abusive interpolations: in these matters, analogies quickly show their limits. However, ideas such as the idea of a limit (in a category) or the idea of adjunction have an eidetic aura which can hardly be restricted to the domain of categorical techniques alone. To talk about them in detail would require the introduction of an arsenal of definitions and sometimes difficult mathematical arguments, which would not be in the spirit of this book. So we will have to restrict ourselves to a few brief indications.

To speak only of it, adjunction is one of the most fascinating and universal mathematical phenomena. It seems to potentially contain both the combinatorics of algebraic theories as well as the essential procedures of transition between theories (for example the passages from topology to algebra), certain principles of symmetry

or, even more surprisingly, the logic of quantifiers. For all these reasons, Lawvere wanted to make it one of the fundamental principles of knowledge:

It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. [64]

It is undoubtedly difficult to follow him that far, but such ambitions are not as excessive as they seem. The essential difficulty lies in the philosophical framework in which to develop such analogies. Seeking to transpose without precaution this or that mathematical concept into a general eidetic context means exposing oneself to the excesses that have accompanied the development of catastrophe theory. We know what this means in the long run: a discredit thrown on the ability of mathematical thinking to inform reality. Transposition is only possible if it is regulated by binding conceptual procedures, prohibiting any unjustified analogy. In concrete terms, this means extracting the eidetic content of a given mathematical construction, i.e. determining and describing the thought processes underlying it, leaving aside the corresponding mathematical results as unnecessary dross.

When the mathematician and the philosopher see spokespersons of catastrophe theory claim that the Aristotelian distinction between power and act can be explained by means of the graph of a singularity or misunderstand the Aristotelian concept of form, they cannot but be dismayed: the method is wrong, and analogical thinking inadequate when it works without regulatory principles.

What I like very much is the Aristotelian conception: the soul as the form of the body [...] The capacity of a body to be the support of a soul is something that is presented as a structure or a law of a formal nature, associated precisely with form in the morphological sense, in the spatio-temporal sense of organisation. The whole is obviously linked to the shape of the flows running through the organism: blood, neuronal, metabolic in general. For me all this is a form, and it comes out of it as a residual form with an organising character: the soul. [81]

Let us explain the dismay resulting of such a reading. If the reference to Aristotle seems, at first glance, accurate, it is in fact based on a series of misunderstandings. In the context in which this reference is formulated (in response to a question which concludes with: “Some who, in the manner of mathematicians, speak of mathematical entities, say that there is a soul...”), the soul is understood in a spiritualist sense (say, in the Christian tradition), whereas the soul is conceived generically in the *De Anima*, to which Thom refers, rather as a simple vital principle. Moreover, the over-determination of the concept of form in a topological sense is unacceptable: if we stick to this concept of form, it is, for Aristotle as for everyone else, the organised physical body which is itself the form of the tissues or flows running through the organism.<sup>6</sup>

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<sup>6</sup> “The soul is substance in the sense of the form of a natural body possessing life in power”. Aristotle repeatedly clarifies the meaning to be attributed to the concept of form in the *De Anima*:

The risks of drift in the use of mathematical analogies are therefore very real. A mathematical result or concept is always multivalent; of course it has a quantifiable content and unconditional validity, but it also most often conveys ideas or intuitions, more or less explicit and susceptible to general (and not necessarily technical) developments—the trick is to master their scope. A common example, passed on in language, is that of the concept of duality which, in addition to a fairly well-determined mathematical content, expresses in its extra-mathematical use the idea of a correspondence between two distinct orders of reality. Thus also of the concept of negative magnitude or inversion, whose common use can reflect, without being aware of it, rather subtle mathematical distinctions!

Lawvere's idea is that the language of categories lends itself particularly well to such transfers of intuitions from the realm of pure mathematics to the field of fundamental philosophical concepts. Let us try to give an elementary but significant example. Thinking about identity and difference is a delicate undertaking.<sup>7</sup> One of the essential difficulties lies in the fact that things that are intrinsically distinct can be said to be identical in a certain respect (the Heraclitean river: a well-defined yet ever-changing entity...). Categorical thinking teaches mathematicians to solve this type of difficulty. It explains to them, for example, that objects rarely exist in isolation. They take on their full meaning when they are inserted into a precise context (usually a "category") and, conversely, it is this insertion that gives them a mathematical status. If two circles are considered apart from the relationships of size and position, it is indeed the same object "circle" which is seen twice in different ways. If, on the other hand, the magnitudes are taken into account, both may differ substantially. Identity and difference in this case are due to the type of relations that determine the identification procedure (the geometric figure alone or accompanied by a notion of size). The Heraclitean river is changing or not depending on what is decided about the status of a certain number of transformations (temporal, physical, ...). The example is worth what it is worth, but it has the value of indication: the frequentation of the mathematical categories accustoms the mind to conceive a certain number of essential eidetic distinctions, sometimes badly identified by the philosophical tradition or of which this one does not know how to give an adequate account.

All these remarks are partly valid for another theory than that of categories: topos theory. Even more than categorical ideas, toposic notions are difficult to explain without a luxury of technical details. Besides, a solid mathematical intuition is necessary to understand what it is all about, so as not to get lost in a rather heavy and indigestible formalism. So, here again, we will content ourselves with indications on a subject which seems to be becoming topical again in the universe of mathematical

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"If the eye were a complete animal, sight would be its soul: this is indeed the substance of the eye, substance in the sense of form" and elsewhere: "The physicist and the dialectician would define anger in very different ways: the latter would see it as a desire for revenge or something similar, the former as a boiling of the blood in the region of the heart, or of the hot element. One gives an account of matter, the other of form or notion." [4]

<sup>7</sup> This is no coincidence, the idea is already to be found in Leibniz's mathematical manuscripts and is part of the general idea of a *Mathesis Universalis* encompassing all orders of knowledge [65].

philosophy. Topoi allow us to illustrate this general idea according to which it is decidedly vain and clumsy to believe that the mathematical universe is closed, and its eidetic limits defined once and for all.<sup>8</sup> The categories teach how to put the notion of transformation at the forefront of fundamental mathematical intuitions. The topoi, for their part, revalue a whole system of topological intuitions: mathematics in the making cannot dispense with regularly looking back at the origin of our intuitions if it wants to be able to create new theories, new mathematical universes. It must also learn to express those of our intuitions which are not yet codified and axiomatized because each one carries within it its own horizon of developments and results.

Topos theory comes from the work of A. Grothendieck around the notion of covering. It all goes back to Riemann's idea on surfaces. Some functions, such as the square root, are poorly defined for complex numbers, since any complex number naturally has two opposite roots. Neither of the two roots can be preferred in general, unlike what happens when only the roots of positive real numbers are involved. Riemann's idea is that this type of function is certainly poorly defined in the complex plane, but that the latter can be replaced by other surfaces which form coverings of the plane. Thus, in the case where the function being studied is the square root, it is possible to be satisfied with a covering with two sheets, which correspond to the two possible roots of a complex number. Regardless of the details, one must simply remember that Riemann's idea is one of the fundamental ideas of the discipline called topology, whose birth certificate it almost marks.

It so happens that the concept of topological space as it was developed at the beginning of the century does not reflect Riemann's ideas very well. Grothendieck, from the 1960s onwards, therefore developed a formalism which made it possible to generalise the definition of what a "topology" is in order to give a faithful account of these phenomena. Basically, this makes it possible to include in the definition of a topology the data associated to coverings. The classical tools of algebraic topology and algebraic geometry fit in this context. In particular, it is possible to speak of local data systems (what mathematicians call sheaves) for these "Grothendieck topologies". These concepts form the basis for the geometric definition of topoi. Nothing simple, as we can see, but Grothendieck had come to conceive them as really essential objects, allowing the synthesis of a whole family of fundamental phenomena.

The notion of topos can still be described in another, mathematically equivalent, but more intuitive way. Set theory forms the basis of the standard axiomatic construction of mathematics. Topos theory can be interpreted as a generalized set theory with topological parameters. Concretely, this means that topos theory is in the same relation to set theory as geometry considered as a whole to... the geometry of a point. Of course, such a statement contains a good deal of provocation and should be taken for what it is: a mere indication. Still, this kind of interpretation of topos theory allows it to intervene in logical questions, such as the construction of unorthodox models of set theory. One of the most famous results of the theory derives from it:

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<sup>8</sup> For a more detailed overview of these issues, see for example Yvon Gauthier's book [34].

the toposic proof of the independence of the continuum hypothesis, an essential and famous result which had first been obtained by more classical methods.

It is becoming more and more obvious, through this type of progress, that intuitionist logic has been too quickly relegated to the dustbin of the history of mathematics. Let us simply recall the judgement of Bourbaki in his historical notes:

The memory of the intuitionist school is undoubtedly destined to survive only as a historical curiosity (N. Bourbaki, *Théorie des Ensembles*, 1970 [11]).

The logical perspectives that emerge from the achievements of topos theory show that intuitionism intervenes in all fundamental logical processes, since it allows us to confront the problem of the finiteness of mathematical construction processes. Thus even a logical argument considered as classical (as the first proof of independence of the continuum hypothesis) is for a good part intrinsically “intuitionist”! Technically, this preponderance of intuitionist logic manifests itself in topoi through their topological or geometrical constitution, which imposes non-Boolean logical calculation rules, that is to say in this case, without the law of the excluded middle.

Technique must not lead astray. Basically, all these results lead back to an essential truth: mathematics rests at its core on a ground of elementary concepts that are probably elusive in their fullness. The finiteness of thought processes is one of the rules governing the use of these concepts. Gödel having been able to show that it would remain, in one form or another, always inescapable, we must accept it resolutely. The joys and misfortunes of the logical programs of categorical or toposic inspiration show how difficult it is to advance in these matters if one sticks to a strictly technical point of view. For all that, it is necessary to encourage these attempts and, even more so, the setting up of a philosophical hermeneutics of logical systems, which knows how to say what meanings the researches propose to reach each time.

Such a hermeneutic is indispensable, since contemporary mathematical logic has yet to establish the legitimacy of its research among the general mathematical public. The posterity of Bourbaki and the post-war schools is still essentially indifferent to all questions of foundation, because it is an attitude which has been methodically taught (at least in France). Logic is almost absent from the teaching of mathematics, and its few appearances are derisory or untimely.

Even beyond logic, one would look in vain for a grand project: to dream today, as Hilbert or Bourbaki did, of the unity of mathematics would have little coherence and meaning, in the absence of an adequate epistemological conception to which this dream would be subordinated. Epistemological questions have been relegated outside the field of activity of “mathematicians at work”, leading to cultural and moral sclerosis. Even within each particular discipline, the situation is confused. A methodical anarchism prevails most of the time, which shows great creativity but also an absence of structuring projects.

The way in which this crisis, hopefully temporary, will be resolved is highly uncertain. The necessary changes cannot be made “from within”. Although technical progress will largely determine the overall pace of development of mathematical

knowledge in the coming decades, a number of epistemological decisions will have to be made. We have had the opportunity to evoke a certain number of them. Among the most significant are the status of logic and fundamental problems, the teleological objectives of mathematics as a constituted science (aspiration to unity?), the necessity of a regulation and an ordering of the acquired knowledge, or the decisions to be taken regarding the thematic priorities.

At the end of these few indications, it should be clear that mathematics should be given a new spiritual life that transcends the technical aspects alone. The chapters that follow will allow us to come into contact with two great mathematicians, both of them outstanding in different capacities, and having had a profound influence on their mathematical environment. It is by no means a question of claiming that their conceptions could in any way fill by themselves the spiritual and organic void just mentioned, but rather of indicating the paths they have chosen to follow, partly on the fringes of a constituted science whose shortcomings they have denounced or whose rules and ambitions they have wanted to modify.

## Chapter 8

# The Homes of Thought



**Abstract** The work of Alexander Grothendieck has opened new horizons to the theory of knowledge. Grothendieck expressed his views on science in texts that are more literary than directly philosophical—if one accepts that philosophical thought must be characterised by systematic organisation and anchoring in a tradition and terminology. We propose here an interpretation of his thought.

**Keywords** Heidegger · Grothendieck · Rêcoltes et semailles

This chapter deals with the horizons opened to the theory of knowledge through the work of Alexander Grothendieck. Grothendieck has had the opportunity to express his views on science in texts that are more literary and “inspired” than directly philosophical—if one accepts that philosophical thought must be characterised by systematic organisation and anchoring in a tradition and terminology. The following lines are therefore very consciously the result of an interpretation rather than a direct reflection of his thought. Nevertheless, this interpretation seems to us faithful to the epistemological choices he made, consciously or not, during his mathematical career.

It is perhaps appropriate first of all to describe his scientific personality in a few words because, although he was the most outstanding mathematician of his generation, he remains rather poorly known outside the scientific community. We will first follow the main lines of the succinct but faithful testimony of Laurent Schwartz, who accompanied his first steps in mathematics, and will comment on them in the light of Grothendieck’s autobiographical writings.

In passing, those who have read Schwartz’s autobiography [77] may have been surprised that, although mathematics has, in his own words, “filled his life”, his book ultimately contains so few prospective reflections or general views on its meaning or future, apart from a few technical and historical pages on distributions and many anecdotes or personal impressions, which it would be very difficult to subordinate to an organised conception of science. An autobiography can very legitimately be factual rather than spiritual, but it is quite surprising that scientists now increasingly favour narrative over reflection in this type of work. The same remark applies to



A. Weil's *The Apprenticeship of a Mathematician* [85], however remarkable Weil's stylistic ease. All this reflects a state of mind about scientific life and its meaning for the general public. Perhaps there is also behind these literary choices the idea that scientific knowledge is not a matter of opinion and should not, as such, be the subject of non-formalised texts? A. Weil says this very explicitly, on the occasion of analogies that Hilbert would have put forward without giving them any theoretical justification, in a letter whose private nature does not necessarily excuse the tone:

The unwritten laws of modern mathematics forbid indeed the publication of metaphysical views of this species. This is undoubtedly better; otherwise one would be burdened with articles that are stupider, if not more useless, than all those that now clutter our periodicals. (A. Weil, Letter to Simone Weil, 1940 [84])

These choices are questionable in any case, as the example of Grothendieck shows, whose genius was able to point out so many new directions for research without respecting the unwritten and somewhat narrow legislation mentioned by Weil. The greatest mathematical advances are the fruit of projects, analogies, uncertain ambitions, bets. Playing the game of a standardised and sanitised science means stifling the creative part of scientific activity. That a balance has to be found is obvious, and Weil is right to insist on a necessary rigour in printed texts. The fact remains that there must be a space where intuition can express its rights. This space remains to be defined, if not conquered.

In Schwartz's testimony [77], Grothendieck's personality is revealed with the excessiveness that is one of his characteristics. In less than a year, he wrote a thesis that became a reference: "a masterpiece of the first magnitude. It had to be read, learned and understood, because everything was difficult and profound". The results are characteristic of what has become the imprint of the Grothendieckian style: the search for the greatest generality, in the service of meaning. Of course, this is one of the fundamental features of the axiomatic programme, but Grothendieck's work goes a step further, because the sense of mathematical constructions has for him an inescapable dimension.

Another fascinating aspect of his personality is the total self-giving of which he was capable in the service of the mathematical work, and which emerges from all testimonies concerning him: "Living alone, working "days" of twenty-five or twenty-six hours, he regularly shifted and reversed, at certain times, the day and night" [77].

This is more than just a detail in the history of his thinking, for it was perhaps in part because of his rejection of the monomania consubstantial with hard scientific work that he eventually turned away from mathematics in favour of a life oriented towards spirituality.

After his initial research on topological vector spaces, Grothendieck worked on geometry and algebraic topology, the methods of which he thoroughly renewed—one of his contributions, topos theory, has already been mentioned in the previous chapter. For his work on the invariants of spaces (Riemann-Roch-Grothendieck theorem) he was awarded the highest mathematical honour, the Fields Medal, in 1966.

His work rises to the level of the great scientific achievements of this century. After 1968-1970, he devoted himself to ecology, and for many years now he has lived secluded in his house in the Hérault, devoting himself to solitary meditation. [77]

In this retreat, he wrote *Récoltes et Semailles*<sup>1</sup> [37], a scientific autobiography on which our interpretation of his thought will be based. This interpretation is furthermore based on a certain familiarity with his published research texts which, although they obey a certain stylistic neutrality imposed by the rules of formalism, nevertheless testify to the depth and scope of his ideas and the originality of his approaches. *Récoltes et Semailles* is an important text, perhaps the only testimony of a major mathematician who is introspective and analyses not only a work, but also its human springs, as well as all the psychological and sociological phenomena involved. Grothendieck describes the conception of science that he has acquired, both at the time when he devoted himself entirely to it, and at the time when he decided to distance himself from it. He returned to mathematics afterwards, his passion for it never having died out, but his work ceased after 1970 to take institutional and agreed forms, remaining at the same time and for long very confidential.

In *Récoltes et Semailles*, against the background of his view of science and his contemporaries, a writing, sometimes passionate, sometimes repetitive, gives a glimpse of his intellectual commitments and the meaning of a life at one time entirely devoted to the study and creation of mathematics. The name of Pascal comes quickly to mind: although his thought has little in common with Grothendieck's, many features bring them together. A common passion for knowledge, a view on science that also knows how to be distant, an attraction for inner meditation. . . Let's listen to Grothendieck: there are many Pascalian accents in the story of his break with the academic world:

It took a shock of great strength to pull me out of an environment in which I was strongly rooted, and from a strongly traced "trajectory"... With hindsight, I realise that beyond the event, there was a deeper force at work within me. It was an intense need for inner renewal. Such a renewal could not be accomplished and continued in the warm atmosphere of the scientific oven of a high standing institution.<sup>2</sup> Behind me, twenty years of intense mathematical creation and inordinate mathematical investment—and, at the same time, twenty long years of spiritual stagnation, in isolation ["en vase clos"]. Without realising it, I was suffocating—it was the sea air that I needed" (A. Grothendieck, op. cit.)

These few lines may be enough to give an idea of the psychological context in which the text was written. The mode of writing is often similar to that of a diary and is surprising: sometimes very controlled, sometimes monotonous, but sometimes deeply moving. The example of Grothendieck shows how artificial it would be to want to think the meaning of science outside of an existential background. Of

<sup>1</sup> My friendly thanks to Ph. Elbaz-Vincent, who provided me with a copy of this text, which had remained in the form of an academic publication and was difficult to access [Trans. Note: when this book was written, as the text has now been published as a book].

<sup>2</sup> The I.H.E.S., of which Grothendieck was a permanent member.

course, there is formal logic, systems of axioms, open problems, and so many other internal phenomena that allow mathematical thinking to evolve in isolation, “en vase clos”. However, they are not sufficient to give it a human meaning.

To understand the springs of a mathematical passion, one must not question the constituted science, but rather the phenomenality of the passion itself—which is, in part, the object of *Récoltes et Semailles*. Grothendieck sees the development of Yin-Yang type oppositions in mathematical work. He thus finds his relationship to mathematics in situations not related to scientific knowledge, where listening or availability to the world find an important place. In this, and however surprising can be the use of the Yin-Yang opposition in this context, he is right, at least on the principle of reducing all forms of thought to universal patterns. Grothendieck sees in these oppositions a dimension of complementarity, which he constantly reminds us of—as if the brutal power of brainwork were the natural and necessary complement to sensitivity and openness to the world. Stopping at the surface of things, irritated also by an atypical discourse, some have seen in the almost mystical language he adopts at times an intellectual bewilderment. However, the spiritualist references of the Grothendieckian discourse correspond to as many original reflections on knowledge in its conflicting proximity to the being-in-the-world.

His distancing from the *cursus honorum* in mathematics at the end of the 1960s was thus accompanied by a systematic questioning of scientific thought patterns, particularly with regard to the individual significance of research work:

For years I had realised that the nature of creative work was generally ignored, obscured by clichés and by ancestral repressions and fears.... This “creative” part of the discovery work is hardly ever reflected in the texts, be it textbooks and other didactic texts, or original articles and dissertations, or oral lectures and seminar presentations etc. (A. Grothendieck, op. cit.)

The idea that governs Grothendieck is not new: the deliberate absence of creative acts in the teaching of mathematics has long been denounced by educationalists. Lakatos, in a rather clumsy way, because he was not a mathematician by profession, had tried to evoke in his *Proofs and Refutations* [62] the hazards of research, sowed with doubts, uncertainties and errors, but his message lacked the strength of conviction that comes from the uninterrupted practice of mathematics. Grothendieck’s testimony is of a completely different nature, since it emanates from a mathematician who can in every respect be considered the Riemann of the twentieth century—both by an obvious thematic proximity to Riemann’s work and by his extraordinary creativity. The authoritative arguments that Dieudonné liked to give in support to his rather rigid theses on the nature of mathematical thought [26] do not hold when confronted to Grothendieck’s testimony.

There has been an extraordinary “flattening”, a “narrowing” of mathematical thought, stripped of an essential dimension, of all its “shadow side”, of the “feminine” side. It is true that by an ancestral tradition, this side of the work of discovery remained to a large extent hidden. (A. Grothendieck, op. cit.)

However, it is difficult to say how this dimension could be expressed in concrete terms. Weil’s argument against “metaphysical views”, which is extremely critical

of informal approaches, is justified by the need to regulate information flows, a problem that is becoming increasingly important given the exponential growth of the scientific corpus.

Is it therefore possible to argue about mathematical thinking outside formal writing? Yes without a doubt, if we believe the testimony given by Grothendieck's unpublished work, mostly written in an indicative way or transmitted by an oral tradition.<sup>3</sup> The lack of logical precision in the details does not invalidate a programmatic idea, the important thing is that it is supported by a solid mathematical culture and organised around an explicit project.

More radically, the research programmes developed by Grothendieck have always had a much higher legitimacy than most mathematical texts published according to academic standards. Whereas the latter often aim at adding a particular result of detail to the panorama of an already constituted science, the former aim only at the most radical intelligibility, the knot of meaning that must be unravelled to access the substance of phenomena. The endeavour is aesthetic, in the sense that it aims to reach the right concept, with all that this means of uncertainties, rather than a determined result. According to the Grothendieckian mathematical philosophy, the results follow from the right statement of the concepts, a bit in the Leibnizian way. Once the "right framework" has been put in place, the questions that had remained difficult to reach until then see their solution impose itself, having lost the somewhat artificial character they may have had before.

Beyond the technical details, which are often rather indigestible, Grothendieck's work is therefore intended as a hymn to the meaning and hidden beauty of concepts. In this sense, he displays a resolute idealism, putting the logical apparatus at the service of a thought in search of the right notion. Situated on the fringe of orthodox conceptions, his description of the processes of creation often has an astonishing tone in a scientific context, but very seductive:

In my spontaneous approach to the discovery of things, the "basic tone" is "yin", "feminine"; and also and above all, contrary to what happens most often, I have remained faithful to this original nature within me, without ever inflecting or correcting it to conform to the dominant values in honour in the surrounding environment.

If I have been able to give my full measure in my mathematical work, and to produce and give birth to a vast, powerful and fertile work and vision, it is to nothing else than this fidelity that I owe it; to this absence of any concern to conform to standards, thanks to which I abandon myself with complete confidence to the impulse of original knowledge. (A. Grothendieck, *op. cit.*)

What Grothendieck claims is the right of mathematicians to think, in the most radical and Heideggerian sense of the word. The mathematical universe is a world in its own right, which sounds of various meanings, of harmonies perceptible but

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<sup>3</sup> [Transl. note] This idea of an oral tradition refers to the late 1980s, when the author of this volume learned verbally about Grothendieck's ideas and research programme. Excepted in specific domains, Grothendieck was overall then much less present in mathematics in the making than he is nowadays and his overwhelming role in the building of a new, categorically oriented, mathematics, was not properly acknowledged.

not necessarily codified. That it is possible to listen to them, therefore in a receptive position rather than in the conquering and brutal dynamic that consists in doing violence to things and concepts in order to “make them talk” can be surprising. Yet this “Yin” attitude makes sense; Grothendieck strives to make it heard, even if such an undertaking is more poetic than merely rational. No mathematician at heart can be insensitive to his testimony, which opens up radically new horizons for mathematical thought by inserting it into the sphere of influence of the thoughts of the being-in-the-world. Perhaps the few words that follow best express the deep aspiration to *say* rather than to *solve* that animated his work: “My lifelong ambition as a mathematician, or rather my joy and passion, has been constantly to discover the obvious. . .” (A. Grothendieck, op. cit.).

A few words in passing against all the misunderstood attempts at popularisation, which believe that they can precisely “say” mathematics by means of ordinary language. Let us be clear, no such thing is the case with Grothendieck, and no serious mathematician would lend himself to this game, except in exceptional situations where, without pretending to be right, an indication may be enough to give a glimpse of “what it is all about”. The obviousness mentioned here is an obviousness acquired at the price of a long familiarity with a corpus and with objects with a complex structure. It comes at the price of hard work, where the Idea gradually takes shape—and takes shape within a conceptual horizon. Pretending to lead contemporary mathematical creation back to illusory elementary intuitions or to perceptions anchored in the lived world is, in its naive form, an undertaking doomed to failure.

It remains to be seen how this conception of mathematics is translated into action. Grothendieck, trained in part in the Bourbaki school, recognises the importance of the idea of structure, but gives it back its programmatic function.

This means that if there is one thing in mathematics that fascinates me more than anything else, it is neither “number” nor “greatness”, but always form. And among the thousand and one faces that form chooses to reveal itself to us, the one that has fascinated me more than any other and continues to fascinate me is the structure hidden in mathematical things. (A. Grothendieck, op. cit.)

The vision and perception of essences play a fundamental role in his mathematical practice. A reference to the beauty and harmony of mathematical beings animates his texts, ignoring modern mathematical philosophies, which have unlearned Aesthetics.

The significance of the Grothendieckian discourse on the participation of beauty in the understanding of things and in mathematical creation will be clearer when explicitly confronted with the Heideggerian texts on the essence of thought. A few excerpts, interpreted very freely in a mathematical context, will serve as a guide.

Few men are sufficiently trained to distinguish a learned object from a thought thing. (M. Heidegger, [43])

In mathematics, the learned object is the object in all the blind violence of its axiomatic definition, stripped of all roots in human thought and its procession of affects, uncertainties and troubled hopes. The thought thing is that which unfolds in

an intuition, that whose presence inhabits and brings to life the work of research. The distance from one to the other is difficult to grasp outside of an experience: it is necessary to feel the presence of the thought thing in order to understand its specificity, to feel a sense where before there was only an inert certainty. From the ability to carry out a logical reasoning to an authentic mathematical thought, the distance is immeasurable and largely unspeakable: the presence of things can never be said without first being endured.

The quality of the researcher's inventiveness and imagination lies in the quality of his attention to the voice of things. For the things of the Universe never tire of speaking about themselves and revealing themselves to those who care to hear. (A. Grothendieck, op cit)

If the courage of thought comes from a call of the Being, then what is given to us finds its language.

As soon as we have the thing in front of our eyes and our heart is listening, tending towards the verb, the thought succeeds. (M Heidegger, op. cit.)

Grothendieck and Heidegger address the same criticism to what is usually described as thought and is most often reduced to the uncritical manipulation of already established notions. In mathematics, thinking means first of all bringing intuition to the concept, which requires prior questioning and openness to things. The axiomatic method teaches how to identify the nuclei of meaning within already established domains, it allows substantial progress, but never authentic creation. Thought itself moves beyond the constituted concepts and brings the experience of things to the word. What it says then is indeed, in its own way, a core of meaning, but its mode of constitution is radically different from that prescribed by the axiomatic method. As Grothendieck puts it in pictorial terms, the mind can choose to live in a house already built by previous generations and seek to repaint the walls or add a verandah to it. This concern for cosy comfort, this more or less laborious do-it-yourself is the essence of academic mathematics. But it can also choose a virgin space and slowly build its own home there.<sup>4</sup> Of course, "he who thinks greatly, must wander greatly" (M. Heidegger. op. cit.), and thinking runs a great and beautiful risk in this game—the very one that Hermann Weyl had chosen to run. Such metaphors are frequent in *Récoltes et Semailles*. That the house of habitation overlaps with one of the central themes of Heidegger's work, that of the "houses of thought", is by no means gratuitous.

Another metaphor characterises the Grothendieckian strategy of mathematical problem solving. Two extreme methods oppose each other when it comes to demonstrating a conjectural result. The first, violent, is that of "the nut and the hammer". The nut represents the question to be solved, the kernels the final result (with its uncertainty content, the nut may be empty or spoiled), while the shell represents the obstacles on the way to the solution. The method consists of hitting the nut with the hammer until it breaks. The elegance of the process is questionable, but it works if the problem is reasonably configured, that is, if it fits well within a certain field of knowledge already acquired. Everything depends essentially on the

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<sup>4</sup> The image is of Grothendieck himself.

quality of the hammer's steel (i.e. the mathematical tools already available), and the hardness of the nut. All the problems that students face, from primary school to the beginning of postgraduate studies, are designed for the use of a hammer. This aspect of teaching probably plays a large part in the rather ungrateful perception that most schoolchildren have of the mathematical science. . .

The other method is illustrated by the rising tide. A compact block of earth plays now the role of the nut. The sea, depicting the spirit at work, covers, as it rises, and gradually dissolves the clay: the problem gradually loses its difficulty as the understanding of its eidetic springs progresses. There is no apparent violence here, since it is the very content of the problem that is modified, and therefore its compactness.

We never reach thoughts. They come to us.

It is then the time for dialogue. (M. Heidegger. *op. cit*)

This second method is less characterised by brilliant and spectacular demonstrations than by patient work of gradual alteration of concepts, erosion of difficulties. It is the one that Grothendieck claims;<sup>5</sup> reading his work cannot prove him wrong.

Those who have not practised research may be surprised by such a description of mathematical work, work of listening and waiting as much as systematic logical effort. However, from the first contact with mathematical thinking, whether it is counting or measuring, similar phenomena occur. Yet they must be perceived as such and one should not dissociate the "poetic" aspect of mathematical activity—the observation, dear to Weyl, of the phenomena of symmetry in nature, for example—from the purely cerebral and mechanical aspect.

This character of thought, that it is the work of a poet, is still veiled.

Wherever it is seen, it is long held to be the utopia of a half-poetic spirit.

But the poetry that thinks is in truth the topology of Being.

To the latter it says the place where it unfolds. (M.Heidegger. *op. cit.*)

The poetry that thinks is the topology of Being: thinking is anchored in the varied of meanings of Being. It must learn to say the "topology" of things, that is to say the structure of their appearance. To what extent can mathematical thinking be thinking in this sense? Our reading of Grothendieck so far would lead to believe that there is little reason to dissociate mathematics from the discourse on Being in all its generality. However, his thought is more complex. It almost comes to nourish the Heideggerian scepticism towards a science which remains apart from the real world even when it knows how to listen to things. The mathematical universe is closed, and its attendance means isolation and sclerosis: the world of mathematical objects certainly has a reality of its own, but it is transverse to that in which human life unfolds.

That is why the passion for mathematics has not been a maturing force in my life, and I doubt that such a passion can foster maturation in anyone. (A. Grothendieck, *op. cit.*)

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<sup>5</sup> But it is characteristic of other great mathematical works, such as that of H. Cartan.

Such a harsh judgement on the existential emptiness of the scientific impulse towards knowledge is all the more surprising because it is authorised by concrete experience. Negative judgements on science and, incidentally, on the “world of technique” usually emanate from thinkers outside this world (such as Heidegger), and whose opinions therefore remain questionable. But for the greatest mathematician of his generation to cast such a critical and bitter gaze has a completely different significance. This is not the place to decide to what extent he is right: in any case, the problem of the existential meaning of mathematical activity is there, and he is to be congratulated for having managed, on this subject as on so many others, to end a taboo. Moreover, his thinking is too fine, too dialectical, to be satisfied with definitive judgements: even though he denounces its limits, Grothendieck acknowledges that mathematical passion has never ceased to inhabit him and to be a source of joy for him. At most he tries to measure these joys by a yardstick that is not that of naive self-satisfaction.

I wondered about the meaning of this stubborn persistence of mathematical passion in my life. When I follow it, it doesn't really fill my life. It gives joys, and it gives satisfactions, but it is not by itself capable of giving true fulfilment, plenitude.

This activity is so fragmentary, it uses only such a tiny part of our faculties of intuition and sensitivity, that these are blunted by not serving. (A. Grothendieck, *op. cit.*)

Grothendieck's case is of course a special one: self-giving without reservation to science remains a rather isolated situation, and moreover very often rather histrionic or neurotic—which is probably the case with most attitudes of self-giving. Most mathematicians succeed in finding a balance between their scientific activity and the rest of their lives, if only by investing the field of mathematics with ambitions that are more mundane than intellectual: social success, power games... However, it remains true that pure intellectual activity often remains what makes the meaning of their life and legitimizes their existence, just as a well-made piece of furniture legitimizes a craftsman's pride and love of his trade. The warning that Grothendieck addresses them touches on the springs of this legitimacy, by inviting them not to be satisfied with a sclerosing autonomy: science will only cease to be riddled with existential conflicts the day it is reconciled with the real world.

Which way forward? The dialectical communion of Yin and Yang, intuition and axiomatic and logical thinking, listening and conceptual violence? All this remains very theoretical. The human action takes its meaning for a good part in its relation to others, and the research is also subject to this law:

When I do maths, I believe that somewhere in me it is understood that it is meant to be communicated to others, to be part of something larger to which I contribute, something that is by no means of an individual nature. (A. Grothendieck, *op. cit.*)

This thing meant to be communicated is the knowledge of mathematical beings. It is first of all the work of a small community: a small group of mathematicians for each given research subject, sometimes only two or three people. But this knowledge overflows from this restricted framework. It is virtually the fact of the whole of humanity and is therefore to be understood in terms of a collective and historical undertaking. While on an individual scale the activity of the isolated mathematician



is horizonless, it becomes significant when measured in terms of intersubjectivity. But the dialectic of the individual and the collective is difficult. The mathematically-loving individual may continue to experience a lack of fulfilment, despite the eventual awareness of a collective mission: Grothendieck says the difficulty, but does not claim to find a solution to it.

The effects of intersubjectivity on mathematical activity are more insidious than they might appear at first sight. Although this activity is largely individual, the simple existence of others never ceases to intervene as an inescapable reference. The modalities of this intervention undoubtedly differ according to temperaments and characters. Such a mathematician will pursue the recognition. Another, more altruistic, will devote himself to the community by seeking to communicate the knowledge acquired. Bourbaki has distinguished himself in this last task, whose noble and generous nature must be repeated. But it is in the very relationship to discovery that these modalities can take on unsuspected traits. The beauty or the interest of a mathematical object could seem to be intrinsic characteristics, and yet it is not so:

The unknown which I pursue in mathematics, in order for it to attract me with force, must not only be unknown to me, but unknown to all. (A. Grothendieck, op. cit.)

Fascination for an object or an idea would therefore depend on collective knowledge and a result, once acquired, would lose its attraction forever. Thus, what is already written in books is no longer unknown, even if we don't know about it: it is part of this uninteresting "heritage". Grothendieck goes even further, and admits that he is not attracted by reading mathematical texts: it is something he has "avoided whenever he could". These are individual character traits: another mathematician will rejoice in the beauty of things discovered by others. More surprisingly, Grothendieck sees here the effects of desire. The object to be discovered is desirable because it is uncertain: let others know about it, and what makes the game valuable is lost. This part of play is consubstantial with the receptive and inquisitive attitude that the mathematician must adopt towards things: Doesn't the "desire of the child fascinated by play" express one of the primary forms of encounter and discovery of the world?

Innocence is another recurring theme in Grothendickian writing. The child marvels at the contact with a world that he approaches without prejudice or rigid conceptual frameworks. It is necessary to know how to listen, but this listening would often remain powerless if it were not accompanied by a certain naivety. To question without worrying about virtuosity or power, to ask without prejudices mathematical objects what their elementary nature is: all these aptitudes would be at the origin of true discoveries. All this can make people smile, and will undoubtedly make those who have had the opportunity to take any scientific journal in hand do it: almost all mathematical work is carried out according to standards other than those of innocence or play... However, the message Grothendieck tries to convey is not without accuracy: one must sometimes know how to break with the appearance of things, and this can only be accomplished by accepting to question the axioms of

a knowledge rigidified around some fundamental concepts.<sup>6</sup> It is then necessary to see “with children’s eyes”, and this look will tell at once what so many fundamental mathematical constructions are artificial!

As far as play is concerned, it can take very concrete forms. Accepting the game content of scientific activity also means accepting to bet and lose! Imre Lakatos has written all his *Proofs and Refutations* [62] on the thematisation of mathematical history as a hypothetical-deductive process marred by errors, which are an integral part of the process. Grothendieck adds to this theme the theme of play, and thus renews it. Lakatos’ book is quite an irritating read for a mathematician by profession. The general idea is good: to show that science progresses by successive approximations, by attempts as well as by genius traits. But the form is detestable. To make students talk about polyhedrons and to make conjectures and proofs emerge from the discussion, as it does, is to ignore what is the essence of research work: an individual and conceptual effort organised by theoretical aims and not only by circumstantial remarks. Lakatos has missed the fundamental mechanisms of scientific discovery. Grothendieck’s testimony has a completely different value. He too recognises that the paths of knowledge are uncertain, but the path he describes from hypothesis to result is organised around the concepts structuring mathematical objects and their perception.

Let us follow him. When curiosity is aroused about something, it must be questioned, regardless of the primary relevance of the question. Even if this question were too naive, or even outright inconsistent, it would still reflect our relationship to the thing under consideration, and would therefore not be devoid of relevance. Whatever happens, the answer we will give will make us progress, and if our first intuition was entirely wrong, we will at least have corrected it. Refraining from asking questions for fear of ridicule is the original sin of scientific education: once one has unlearned to question, the paths of creation are restricted to those already traced by others. More often than not, a mathematical question has the form of an affirmation—a conjecture. And, in general, the first conjecture is false, because we have only just come into contact with an object and our understanding of it is too crude to serve as a faithful guide. What does it matter! Perhaps it will be enough to write the conjecture in black and white for its falsity to be obvious, but “before writing it down there was a blur, like an uneasiness, instead of this obviousness”. Rich to know our first conjecture false, we come back to the problem less clumsy. Thus, step by step, our intuition is enriched, learning to better define the exact contours of the phenomenon. Errors have enamelled this process, but they are the ones that have allowed us to progress: to do without them, we would have to limit ourselves to mathematics without charm, because it would be devoid of unexpected events. But it is also possible that the progression itself may lead to a major misunderstanding: as the work progresses, minute distortions emerge and

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<sup>6</sup> Thus, Grothendieck has often questioned the axioms of topological theories, which he believes do not adequately reflect our intuitions.

then get understood as such. The tension rises, until the moment when the error bursts, causing the collapse of a certain perception of things:

The discovery of error is one of the crucial moments, a creative moment among all, in any work of discovery. . . . It is a moment when our knowledge of the thing being probed suddenly renews itself.

When we are moved, not by the fear of seeing an illusory security vanish, but by a thirst for knowledge, then error, like suffering or sadness, passes through us without ever freezing, and the trace of its passage is a renewed knowledge. (A. Grothendieck, op. cit.)

These traces never appear in mathematical texts: both the dream and error parts are always erased as useless residues. Error is undoubtedly difficult to present in an interesting way. In any case, one must experience its necessity individually in order to experience its share of truth: to describe an error, it loses all its pedagogical value.

Making mathematical research understandable cannot be done in mathematical language; this is the business of the philosophy of science or of poetics such as that inaugurated by Grothendieck—access to which presupposes a great familiarity with the intellectual processes at work in science. It would be nice to be able to give an account of the essential phenomena of thought: curiosity, interest, sense of beauty, intentionality, receptivity, etc. All these terms are tragically absent from contemporary mathematical thinking. Grothendieck's is sometimes surprisingly atypical; we have tried not to distort his views, using as much as possible a vocabulary close to his (Yin-Yang, receptivity, play. . .). This fidelity is also an expression of deep respect. Approaching *Récoltes et Semailles* is a difficult and delicate undertaking, but the text speaks for itself and tries to say a mathematical thought, where other autobiographies or mathematical confessions often turn out to be mere collections of details and anecdotes.

On arrival, the impression emerges that Grothendieckian thinking, as it has just been outlined, can bring original concepts and, even more so, a new tone. Descriptive mathematical philosophy in the manner of Brunschvicg [13] was historically important, but it is no longer sufficient today. Talking about concepts is useful, but we must also say the thought that organises them: a thought that sees, believes, plays, a living thought, and sometimes charged with humanity.

## Chapter 9

# To the Meeting of Reality



**Abstract** Contemporary mathematics is convinced of the decisive role played by the physical, life and social sciences. At the same time, the quarrels between pure and applied mathematics are brought to lose their importance.

**Keywords** Analogy · Applied mathematics · Thom

With respect to the meaning one should attach to mathematics and its connection with the world of senses, Bourbaki has never tried to express publicly any opinion, leaving everybody free to hold his own views, as long as his mathematical writings conform to the rules of logic which are generally accepted i.e. those of Zermelo-Fraenkel. (Dieudonné, [27])

The apparent neutrality of the text and the asserted will to privilege the mathematicians' "freedom" of epistemological choice should not deceive; in fact, they reflect a profound indifference to the effectiveness of mathematical discourse. This indifference, whose untimeliness in the twentieth century is manifest, has had dramatic consequences. Bourbaki, and with him a whole school of thought, thus "missed" mathematical physics, one of the fundamental mathematical fields, and one of those whose advances will remain in history. Let's be clear: it is not a question here of making a new trial of intentions to Bourbaki, but of regretting the attitude of a part of the mathematical community, to which he was able to serve as an intellectual guarantee. Bourbaki himself was not entirely unanimous on this point. P. Cartier, whose role and activity within the group is well known, never ceased to militate in favour of the recognition of the importance of mathematical physics and, beyond that, in favour of disciplines essential to contemporary mathematics, but which Bourbaki was not interested in: let us quote combinatorics or probability.

Anything related to mathematical physics is totally absent from Bourbaki's text. I contributed to the Bourbaki seminar with a series of articles focusing on mathematical physics issues. But for a long time I was the only one, and my contributions were not always accepted without a fight. (P. Cartier, [15])

A. Weil may well embody the opposite attitude. An anecdote, reported by P. Cartier in the same article on the life and death of Bourbaki, illustrates better than

any other commentary a certain vaguely sufficient condescension of Weil towards the “mathematics of reality”:

A few years ago, in a discussion with André Weil, just after he had published his memoirs, I said to him “you mentioned that in 1926 you were in Göttingen. . . something happened in 1926 in Göttingen.” And Weil asked me “What happened in Göttingen?”, and I said “Well! Quantum mechanics!” And Weil replied, “I don’t know what it is. (P. Cartier, op. cit.).

Weil’s provocations or Dieudonné’s indifference are now from another age. Contemporary mathematics is convinced of the decisive driving role played by the physical and life sciences, as well as by the sciences oriented towards “social physics” (game theory, economy, finance, . . .). It resolutely accepts to open up to the world, to take in charge the organisation of the dialogue with the other sciences—something that Auguste Comte will no doubt be delighted about! At the same time, the quarrels between pure and applied mathematics should be brought to lose their importance. To say that they are entirely outdated would be to go too quickly to the task, but mutations are observable, transverse to the traditional partitions. Perhaps applicability will soon cease to be a demarcation point and give way to other conceptual breakdowns. The difference could be made between theoretical mathematics and engineering mathematics, with all the nuance bearing on the very nature of the mathematical work and its degree of eidetic content, rather than on its effectiveness.

There is another concept than applicability (or effectiveness) to serve as a criterion for a whole area of mathematical activity. It is the notion of modelling. We owe to G. Israel to have underlined its importance in his book on the mathematisation of reality [53]. In traditional mathematical thought, the mathematical expression of a given situation translates its physical essence: the setting in equation coincides, so to speak, with the individuation of the substance of the phenomena. One might thus have thought that it would be enough to push the calculation far enough to determine the fundamental structures of the world—this is the postulate of mechanism, which for a moment was able to impose itself well beyond the sole domain of mechanics. The limits to the corresponding programme have multiplied, limits of principle first of all, but above all technical limits. Mathematisation never adheres sufficiently to reality for the latter to restrict itself to its mathematical model. It is enough to push the calculation far enough to reveal unsuspected constraints that challenge the relevance of brutally putting phenomena into equations.

The better understanding of turbulence phenomena, chaotic processes and, more radically, the relevance of uncertainty calculations in quantum mechanics, have brought mathematicians a great deal. And this perhaps less in terms of mathematical content and results than in terms of their perception of reality and the field of activity open to them. Technical barriers limit the power of mathematical apprehension of phenomena, but they are also invitations to search for new strategies. Since the world does not lend itself to the game of already elaborated calculations, it is necessary to go ahead and renew the approaches. There are several ways to do this. First of all, there is a sectorial solution: each type of phenomenon has its own logic, and therefore a mathematics is subordinated to it. It has thus been possible to witness

the emergence of biomathematics or financial mathematics, whose situation and autonomy with regard to “mathematical science” is not yet well clarified. It is also conceivable to seek to codify, as a whole, these new strategies of approaches to reality. René Thom has tried, more and better than any other, to do so.<sup>1</sup> Even if his ideas are highly debatable, distorted as they are by a certain obstinacy to focus on catastrophe theory, they can nevertheless serve as a spur to thoughtful research on the very meaning and scope of mathematical modelling.

First of all, it is a question of agreeing on the very notion of a model. Restricting the content of a mathematical concept to its structural or axiomatic component is equivalent to making incomprehensible its possible hold on reality. Therefore, the first step is to break with the myth of a purely ideal and transcendent mathematics. Whatever their participation in formal and “eternal” Ideas, mathematics also has material functions and sensitive origins (aesthetic ones, in the Kantian sense). Thus, the idea, dear to Thom, of boundary is abstract from our frequentation of the objects of the lived space. The notion of boundary,<sup>2</sup> as much in the intuitive sense as in the sense of differential geometry or topology, manifests itself to us through sight or physical contact. An exchange occurs in two directions: perceiving the abstract notion of boundary in its concrete manifestations enriches our mathematical intuition, but in return our perception of the lived experience is refined by developments in mathematical theory: mathematical thought inhabits the real world and informs perceptions!

Taking this dimension into account must change the common perception of mathematics. Even if it is not capable of exhausting the content of reality, or of converting it entirely into formulas, it is capable of structuring portions of it. The case of economics or finance is exemplary: mathematical models have only a very relative validity, and their theoretical justifications are at least suspect. Nevertheless, mathematisation provides economics and finance with tools of analysis and a better apprehension of the phenomena.

It is in a form of mathematisation of reality with a strong qualitative and topological component (as opposed to reductionist and highly quantitative approaches) that Thom and his disciples embarked on with the theory of catastrophes, with more enthusiasm than success. Although they largely failed, the endeavour deserves to be continued, and the global and structural approach to phenomena as Thom envisaged it for biology must remain a long-term source of inspiration for mathematicians.<sup>3</sup>

The starting idea for his work is topological in origin. R. Thom is undoubtedly one of the great names in topology in the 1950s.

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<sup>1</sup> Among others in [79].

<sup>2</sup> The mathematical notion of boundary generalises the usual notion: a circle is the boundary of a disc, a sphere of a ball. The corresponding fundamental algebraic theory, known as cobordism, was developed by Thom, who was awarded the Fields Medal for it in 1958.

<sup>3</sup> An example of a successful approach to biological phenomena by an exceptional mathematician is given by the work of the Ukrainian mathematician Israil Moisevic Gelfand (1913–2009) on the principles of organisation of complex multicellular systems.

So [after the Fields Medal, awarded in 1958 for the theory of cobordism], I built up a kind of semi-philosophy. This is how I characterise catastrophe theory. Some people have said that it is bad science coupled with bad philosophy... Perhaps they are right. But it seems to me that it is something quite original, and finally quite valid. (R. Thom, [81])

This semi-philosophy is intimately linked to his mathematical work. One of his guiding ideas is that the notion of boundary is among the most universal of all, somewhat like the great philosophical categories or the Aristotelian *post-praedicamenta*. We distinguish objects by their contact surface: the discontinuity between their interior and exterior is the phenomenon that marks their presence in the world.

The whole “Thomistic” justification of the philosophical importance of the mathematical concept of boundary lies in this remark, and one can understand the fairly legitimate mistrust that may have been exercised from the outset. Especially since Thom goes further: he links his works to the Aristotelian conception of Being, and especially to the distinction between form and matter, and its various avatars.<sup>4</sup> The conceptual drift and the philosophical overkill that surrounded the theory of catastrophes, culminating in these debatable references to Aristotle, have ended up discrediting it in the eyes of the greatest number. All the more so as its applications on the scientific side remained clumsy or hypothetical, and that, in its mathematical aspects, the theory gradually turned out to be a theory like any other, with its charms and eidetic limitations, but in any case without any real epistemological specificity.<sup>5</sup>

At one with the idea of a boundary, limit or demarcation, “the catastrophe is permanent without our being aware of it” (R. Thom, *op. cit.*). But also, to be permanent and universal, the catastrophe becomes a hollow concept that only naivety or unbridled imagination can adorn with attractions. Wanting at all costs to make catastrophist thought the core of all intelligibility, it has lost its specificity and interest to become the receptacle of risky speculations. So, with the hindsight provided by twenty years, after having pointed out its intrinsic drifts, we will then abstract from all that is excessive in it, from all its ontological pretensions in particular, to preserve only what continues to be valid today.

Beyond its technical aspects, catastrophe theory wants to have a hermeneutic function, which will be claimed here for all mathematics. The mathematician’s experience feeds his perception of the real world. The meanings that inform Thom’s practice of topology have their counterpart in the lived space, and topology, from abstract theory, thus becomes a tool of interpretation and a kind of qualitative geometry at the service of the apprehension of forms. One of Thom’s mistakes was not to realise that what is true for his topology of singularities and boundaries is true for any other form of mathematics. Every engineer penetrates and informs reality with theoretical data, and the mathematics of the strength of materials will give the bridge builder a view of the girders quite similar to that which the topologist will cast on an evolving form.

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<sup>4</sup> See Chap. 7 of this book for an analysis of Thom’s concept of form.

<sup>5</sup> See [6].

A proponent of the philosophy of catastrophes would reply that there is a notable difference, which makes the originality and scope of his philosophy: the science of the engineer is quantitative, whereas that of the topologist is qualitative. But the argument does not hold, for several reasons, obvious to the mathematician, but which can mislead an uninformed amateur. First of all, make no mistake, topology is just as quantitative as any other science. It aims at calculating numbers, invariants, and is not a simple mysticism of shapes, as we can sometimes believe when reading Thom! To go further, it is the science which has most deliberately pursued a project of quantification of geometric entities: since its birth, with Riemann and especially Poincaré, topology has always wanted first and foremost to succeed in giving a quantitative characterization of shapes or, to put it another way, to digitize the qualitative! From the outset, Poincaré had very concrete applications in mind when he drew up the first foundations.<sup>6</sup> When Thom claims to substitute the search for quantitative information with that for qualitative data, he takes advantage of the little information available to most of his readers about the nature of the topological enterprise. In fact, he simply proposes to juxtapose to the usual forms of mathematisation of reality a new form, where the calculations would aim at a new type of information in its own way just as quantifiable as any other.

If the catastrophist approach does not therefore differ radically in technical terms from another type of approach to reality, it nevertheless has the merit, in its hermeneutic ambition, of seeking to clarify the links between mathematics and reality. It is in this respect, largely independently of its topological contents, that Thom's philosophy is, from our point of view, the most interesting. A fundamental aspect in the study of the relationship between mathematics and reality is that of technique. Technical aptitude is an instrument of domination and world understanding, but in modern times domination too often obscures world understanding, although both are consubstantial in the essence of technique. The excesses of the twentieth century made us lose sight of this, and yet even before the concept of technique appeared as such, probably in the fifth century BC, this consubstantiality was already clearly perceived. We would therefore be tempted to assert that Thom's mathematical hermeneutics has completely different origins from those he attributes to it, and is not dissociable from the general problem of technique.

To take just one example, familiar to the Hellenists: in the *Odyssey* already, technical mastery appears as an essential component of Ulysses' skill and is one with the other traits of his personality, such as oratory or physical skill. It has of course an instrumental function, but also an interpretative one: the work carried out says the skill of the craftsman, the shape of a weapon says the quality of its design. The mistake would be to believe that this type of information is of a different order from that which underlies modelling attempts à la Thom. In both cases: evaluation of the effectiveness of a weapon or of the topological type of a given form, the information obtained is first of all qualitative and emerges from taking into account

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<sup>6</sup> The idea of topology and some of its most spectacular techniques (Leray's theory of spectral sequences) stem from motivations linked to the study of differential equations.



a global situation rather than from a reductionist breakdown of the information and from putting things or phenomena into equations. As for the mathematical aspect of things, it is undoubtedly less decisive than it seems. All technical information about the world can be mathematised, in one form or another, according to a suitable model—i.e. by allowing sufficiently flexible and complex mathematical structures. The classification of objects or concepts into categories or types, for example, is nothing other than reasoning by equivalence classes: in their own way botanical or zoological classifications are just as “mathematical” as a calculation of the resistance of materials. The point is to extend the field of application of mathematics adequately. In the majority of cases, such a mathematisation is useless and redundant, which explains why it is not made explicit—if not by pure pedantry. We all know that one cannot, in general, undo a knot by simply pulling on the ends of the string! It is a mathematical theorem, and it so happens that the mathematical theory of knots is extremely deep and rich in lessons, but it is also a “theorem of practical reason”... One must be aware of the validity and the deep unity of both points of view, this is what we will choose to retain from this first (unorthodox) path in Thom’s thought.

As for the limitations of “Thomism”, beyond the reasons that have already been pointed out, they stem from a most common error of principle. All those who have wanted, in one form or another, to approach philosophy through mathematical concepts have stumbled on it. The project consisting in wanting to make mathematical objects (or a particular class of objects, such as singularities, elementary catastrophes, etc.) as many hermeneutical models is, by nature, doomed to failure. Because any movement of modelling of the real is by essence dialectical: the real must permanently inform the theory, otherwise the theory will freeze in caricatured positions, before feeding vain polemics.

In their defence, the will to make a particular theoretical approach to reality an inescapable paradigm is not peculiar to scientists in search of epistemology! All the petty quarrels that go along with the will to lead everything back to an exclusive principle (and, incidentally, to its inventor) clutter contemporary epistemology with a jumble of neo-scholastic discussions, which would be enough to disgust any sensible person endowed with reason from the love of science. The Anglo-Saxon philosophy of science in particular is full of these detestable quarrels. Luce Giard gives an account of them in her introduction to the French translation of the texts of Lakatos [63]; her description speaks for itself:

From time to time, the great English-speaking family, relatives and allies would get together [...], each clan delegated its best champions to face, in this closed field, the opposing fighters, in front of an informed public of connoisseurs, who counted the blows and the points, the law of the market stating as it went along, without mercy, the quotation of securities on the stock exchange of ideas. At that time, it was appropriate to play Carnap and positivism down, Popper up, and Kuhn, a new stock to follow, on the secondary market.

Returning to modelling, it is subject to intrinsic limitations. As G. Israel pointed out, the new modelling methods are deliberately partial, incomplete and choose to capture only one component of reality among others. The corresponding

hermeneutics must be conscious of being able to reach only a fragment of reality, and be satisfied with this contribution.

What characterises the models is that they show the existence of an analogy. In the case of mechanical models, it is a mechanical analogy; in the case of mathematical models, a mathematical analogy. Now, while the first concept is quite clear (to build a mechanical model imitating the studied phenomenon), the concept of mathematical analogy is less clear. (G. Israel, *op. cit.*)

The concept of analogy would therefore be decisive for the immediate future of mathematical philosophy. From the point of view of the history of philosophy, it should be noted that the concept is largely of mathematical origin, the prototype of the analogical ratio being the ratio of proportionality (if  $x$  is to  $y$  as  $z$  is to  $t$ , there is identity of proportions). Aristotle was able to extend its use to all fields of thought. From Ethics to the theory of Being, everything can be reconducted to analogical reasoning, often through the search for an equilibrium between opposites, which is a good part of the subtle charm of Stagirite thought. Analogy, in the strict sense of an identity of relationships like the one found in geometry, is too restricted to be used outside the mathematical field. Therefore Aristotle specifies the rules of a general use of the concept: analogy designates, by extension, the proportional resemblance, which allows it to appear alongside the metaphor in his Poetics:

I mean by “relation of analogy” all cases where the second term is to the first as well as the fourth to the third, for the poet will use the fourth instead of the second and the second instead of the fourth. (Aristotle, Poetics [5]).

Analogy has recently aroused keen interest, both on the part of philosophical historians, who have been studying medieval theories of analogy, particularly in its onto-theological aspects, and on the part of scientists, who have more or less happily claimed the validity of the scientific use of the analogical method.

We will take a last look at Thom’s case. Let us say it from the outset: if the “Thomist” theory of analogy is hardly convincing, it has once again the merit of pointing out problems central to mathematical epistemology.

The essential interest of this theory [catastrophe theory] is certainly to provide patterns of intelligibility in science and elsewhere. And this seems to me quite valuable. Even if sometimes we are mistaken; even if sometimes reality is governed by an evil genius who proposes pseudo-intelligibilities to us. The main interest of this theory is to propose a mathematical theory of analogy. Analogy is a mental operation which, in principle, has nothing to do with a well-defined substratum. One can apply analogical thinking to very different situations, without worrying about dealing with physics, chemistry, biology, sociology (R. Thom, *op. cit.*)

In a sense, his whole method is gathered, from a philosophical point of view, in these few lines. They reveal, first of all, an understanding of the theme of analogy in accordance with the Aristotelian extension of the framework of the theory of proportions to an almost universal “metaphorical” and functional context. But they also testify to a fatal blindness, which leads to making catastrophe theory the “mathematical counterpart” of the theory of analogy—even though the latter has from scratch a mathematical origin, and its extension from mathematics to a general

theory has already been codified by Aristotle. In other words, R. Thom, when he rediscovers through catastrophes the mathematical scope of analogy, inconsiderately ignores the very content of the concept, which he ends up distorting, whereas it would have been perfectly possible for him to develop his ideas within the framework of Aristotelian theory. But he would then have had to renounce the exemplarity of the concept of catastrophe. . .

We know the examples of analogies given in the *Poetics*:

There is the same relationship between the cup and Dionysus as between the shield and Ares; the poet will therefore say of the cup that it is “the shield of Dionysus” and of the shield that it is “the cup of Ares”. In the same way: there is the same relationship between old age and life as between evening and day; the poet will therefore say of evening, with Empedocles, that it is the “old age of the day”, of old age that it is the “evening of life”. (Aristotle, op. cit.)

The second analogy can have a geometric interpretation:

It is clear that the organising knot of the analogy is this idea of the end of a regime, of the end of a period of time, what we call old age on the one hand and what we call evening on the other. It is this kind of neighbourhood of the terminal moment, of the terminal point that you consider. There is an underlying geometry here which fully explains the content of the analogy. Nothing can be objected to it. (R. Thom, op. cit.)

Although there is nothing to object to, here are some objections to R. Thom. What does Empedocles mean if he makes the evening “the old age of the day”? That the evening is a “tubular neighbourhood” of the night ? Perhaps, but one can also think that he is trying to bear witness to an obscure sensation of elapsed time, of the day that is coming to an end and which is judged by the morning’s projects and hopes, or even of the fatigue of the ploughman returning from the fields, similar to the weariness of an old man.

The example teaches two things. First of all, the metaphorical dimension of analogy is in no way reducible to a mathematical model of any kind. There is no geometry to Empedocles’ writings, in any way, except to mishandle them in a way that is as reductive as it is pathetic. However, trying to give a mathematical model of an analogy is not without meaning. To say that there is a geometrical component in a relationship is to reveal one of its meanings and to progress in its understanding.

The example of the cut-Dionysos, shield-Ares analogy relation speaks for itself. It would be rather difficult to clarify it through the geometry of the catastrophes. . . If we disregard its poetic component, it can however be approached through concepts such as that of association or relation. The symbolic component of the analogy (the cup “symbolizes” Dionysus like the shield Ares) will thus refer to the ideas of G. Frege on denotation. In two words, mathematisation is an open process, both in its methods and in its links to non-mathematical disciplines. It does not have to favour this or that method from the outset, whether it is a question of catastrophes or any other mathematical theory: it is always reality itself that dictates the content of the scientist’s conceptual choices. Some mathematicians would express this constraining character of reality by saying that mathematical work is

more testimony than discovery, more openness and listening than invention and violence—Grothendieck has given us enough information on this subject!

Beyond their own content, rather limited for both technical and methodological reasons, mathematical philosophies such as that of R. Thom, teach us today above all to reconsider the relationship of mathematical thought to its own history as well as to its cultural environment. The greatest philosophical merit of R. Thom himself is undoubtedly to have recognised the importance of Aristotelian thought and its unconditional topicality in the field of mathematics, so long overshadowed by some chronic misunderstandings linked to the failures of medieval Aristotelianism. For the inadequacies of the *Physics* in no way justify the discrediting of the Aristotelian theory of science conveyed by a certain post-Koyrean<sup>7</sup> vulgate! Thom's mistake—and that of his disciples and supporters—was, once again, to gauge Aristotle by the geometry of singularities, an inconsistency that would have been venal if it had not at the same time obscured this essential idea: the relevance of Aristotelianism!

We will conclude this book with a critical and updated rereading of Book M of the *Metaphysics*, trying to suggest a few paths—some “*Holzwege*”—that are still open to the endeavours of mathematical thought—provided that this thought is able to return to its origins, and to questions that have been hidden for too long. The two great questions, which are basically unresolved, and which it still has to face if it wants to be able to give a cultural and humanistic function to mathematics, have hardly changed since antiquity. Does mathematics have a hermeneutic function? What is its ontological status, and more precisely its mode of existence and its relationship to the “real world”? Book M addresses the major problems that have recurrently run through this book. It breaks with the various currents of thought that would like to make mathematics a universe in itself, detached from particular contingencies. By nature, the universal is always inscribed in the particular. I can, for example, draw a circle and deduce from it, by a determined construction, a property which will be valid unconditionally for all the circles and thus for the “circle in itself”. This is the archetype of a mathematical advance, which always proceeds from factual data in order to access the truth. Consciousness is indeed, in its own way, always factual: truth only ever unfolds through limit, ideal or immanent processes (according to the various modalities of its appearance), and it is always obtained from the consciousness of a ‘something’, which is never identifiable with the corresponding mathematical object.<sup>8</sup> Thus the opposition of an eternal truth to the contingency of the real world is vain: mathematical discourse is, like any other discourse, anchored in the lived world. This is also what justifies the very possibility and efficiency of mathematization that build models on the basis of phenomenal data. These models are provided with the logical validity

<sup>7</sup> Alexandre Koyré (1892–1964), French philosopher of science of Russian origin.

<sup>8</sup> This is already the case with the circle, but Husserl showed that the phenomenon was general: it results from his *Psychological Studies for Elementary Logic* [48] that we never have a proper representation or intuition of mathematical objects.

characteristic of mathematics as long as the concepts which structure them are themselves mathematical.

This is how science was able to develop the idea of Brownian motion, based on the observation of chaotic movements on a microscopic scale. All the difficulties of mathematical thought and pedagogy are due to the fact that this movement from reality to abstraction or from existence to the universal can be erased artificially. It is so easy to introduce the Brownian motion in probabilistic terms without leading it back to a concrete experience. . . The book M says well this difficulty, while teaching that the thought of the universal and the particular is one and the same thing:

Excellent results can be achieved in the study of each question by positing as separate what is not separate, as the arithmetician and the geometer precisely do. (Aristotle, *Metaphysics* [3]).

Positing something as separate is to act *as if it exists in itself*. Everything allows us to do so, starting with the fact that, according to the terms of the *Metaphysics*, Being is said in several meanings; but we must keep in mind the rooting of all thought in the phenomenal universe. The mathematical philosophy of this century believed it could transgress this principle, and thus lost all existential reference point. More than anything else, this is evidenced by the inability of the theories of formalising inspiration to account for the feeling of beauty.

The philosophers who claim that the mathematical sciences make no room for either Beauty or Good are certainly mistaken: on the contrary, Beauty is the main object of the reasoning of these sciences and their demonstrations. It is not a reason because they do not name it to say that they do not talk about it, because they show its effects and relationships. (Aristotle, *op. cit.*)

Of course, these are only fragmentary indications, as is in its own way the too brief book M, but they are in the direction of the future of mathematics, that is to say, in the direction of a reconciliation of theoretical activity with the worlds of intuition and experience. Each great philosophical question proceeds from an antinomy. The one that determines the field of mathematical philosophy is undoubtedly the antinomy between the phenomenal and ontic origins of knowledge and their immutability, their transcendence of a logical and formal nature. This antinomy takes shape in familiar situations. Thought goes from the simple to the compound, and thus privileges the point over the line, and the line over the surface when it comes to giving definitions (a line is a set of points such as, etc.). However, the experiment proceeds in the opposite direction: a surface is obtained by a limit process, it is a three-dimensional body whose thickness “is very small” or “tends towards zero”. To speak like Aristotle, logical anteriority does not entail substantial anteriority, and the dialectical movement between the two animates and governs mathematical thought, which sometimes clings to the certainties of calculation, sometimes allows itself to be led by the play of intuitions, listening and receptivity to things.

When Thom wants to transform geometry into hermeneutics, he tends to restore the existential dimension of the mathematical act, going against a certain orthodoxy indifferent to the meanings of action (action is understood here in the theoretical sense, and not as the application of science to practical ends). In their own way,

the ideas of Hermann Weyl or Alexandre Grothendieck go in the same direction, although according to significantly different modalities. In any case, these three mathematicians, to whom we must add Aristotle and, closer to us, Husserl and his theory of science, want to tell us, against all odds, that the logical anteriority of pure forms is not without an ontological posteriority. It is up to mathematicians to take note of this!

# Chapter 10

## Conclusion



**Abstract** Since the establishment of the structuralist method, the most eminent mathematicians have often supported the idea of the independence of mathematical contents from their philosophical implications or presuppositions. We argue that modernity must go in another direction and renew the dialogue. It is up to the mathematicians of the 21st century to rethink the horizons of a science whose contours and possibilities are much less fixed than they seem.

The overview of mathematical thinking that has just been given has no claim to completeness. It has been aimed each time at what seemed to be the essence of an idea or a theory, leaving in the shadows a jumble of details and uncertainties that remain to be clarified. It was also necessary not to be encumbered with technical studies, which could only have convinced a mathematical reader, without changing the substance or the content of our conclusions.

Generally speaking, it is doubtful whether there is a real need to measure oneself against the technique in order to be able to evoke the guidelines of the major cultural and intellectual movements. If this were not the case, they would remain compartmentalized and purely esoteric. The proof of Fermat's theorem has convinced those who still had doubts: if the precise mathematical contents have little intellectual repercussions beyond a few circles of initiates, the overall movement of science continues to be a social phenomenon, and the demonstration of the theorem has had a certain echo, essentially independent of the methods used, that is to say of the very substance of the result. Should we conclude from this that pure science has a value and a role vis-à-vis the collectivity that would be independent of its contents? That there is research, a breeding ground for men of science, collaborations that structure an international community of scientists, would then be more important than the activity deployed within it. If it is not necessary to go to these extremes, the question remains of the topicality and the cultural function of an endeavour like that of mathematical research. This book has tried to show that it is the vector, beyond its tangible and quantifiable successes, of real epistemological debates, which are still far from being decided. The conclusions which will emerge from them will determine, much more than this or that result, its collective meaning.

Other paths to contemporary mathematical discourse were conceivable than the ones we followed. In particular, a philosophical history in the manner of Brunschvicg remains to be written for the twentieth century. But it is not certain that the means are already in place: the necessary hindsight is lacking, as well as a better knowledge by the public of conceptual advances and controversies. Here we have to pay the price of the lack of media and cultural coverage of mathematics. Another difficulty, more radical, lies in the decoupling of technical and eidetic advances: since 1950s, that is to say since the establishment of the structuralist spirit, the most prominent mathematicians have most often endorsed the idea of independence of mathematical contents with regard to their implications or philosophical presuppositions. Science has been deliberately dissociated from its philosophy.

Is this detestable attitude about to come to an end? It is to be hoped so, and it may well come to pass through the new “applicable mathematics”. The ever-increasing links between mathematical research and the phenomenal universe require that the foundations of mathematical validity be rethought, particularly in its function of approximating reality. As this debate is largely independent of the major historical issues of mathematical epistemology, it is reasonable to think that it will be able to free itself from the conceptual burdens which continue to undermine debates on foundations or logic. Once they have finished with epistemological indifference, obvious building sites await mathematicians—all the more obvious as contemporary philosophy expressly points them out to them. Thus, that professional mathematicians should (re)read Husserl with an unbiased view should go without saying, for where else should they go to find the founding materials of this post-structuralism which their science so terribly needs?

It remains to be seen whether these opportunities for regeneration and reform will materialise. The turn of the century, with the year 2000 declared the “World Year of Mathematics” could well have set the tone. The congress of mathematicians in 1900 had thus kicked off the triumph of the axiomatic movement, which was to become the major phenomenon of the century. If the scientific community is seeking to update the message Hilbert sent at the time, the results obtained so far do not seem to live up to expectations: the community of mathematicians lacks too much unanimity, landmarks and emblematic figures for that. In any case, the major problem will remain the one that Hilbert had already glimpsed: the impossible unity of the corpus, the fragmentation into competing disciplines, the loss of a true identity, of a feeling of belonging to a common enterprise.

From a “French” point of view, by admitting that national references still make sense in contemporary mathematics, Bourbaki’s posterity remains to be thought. By taking note of what P. Cartier, and others with him, have called the “death of Bourbaki”, the way is open for a renewal of one of the greatest collective scientific undertakings of the century. The criticisms we have addressed to Bourbaki throughout this work should not conceal a deep attachment to his work and his universalist ambitions. These criticisms were aimed at prospective errors and teleological choices, which history has shown to lead to dead ends: Bourbaki still had to exist and his thought had to be as influential as it was for these conclusions to



emerge! The mathematical universe has evolved too profoundly since the 1930s for the Bourbakist enterprise to be able to regenerate itself to the fullest extent desirable without a radical questioning of its presuppositions: the field of contemporary mathematics is far too wide for it to be ordered by a small group, whatever the quality of its members. As for the idea, consubstantial with bourbakism, of an “architecture of mathematics”, we have seen its intrinsic limits.

The organisation of the corpus (or its reorganisation), the prospective aims, the essential epistemological choices must be the subject of collective work and debate in the future. What mathematics, and all the other forms of knowledge, need the most from now on, is an orientation table, treatises, structuring rather than didactic, fixing the lines of force of the researches already completed, the perspectives of the works in progress, designating the great problems to be faced. This is where the history of science takes on its full meaning: it is responsible for the management and organisation of the accumulated knowledge. But, let’s be clear, such a history of contemporary sciences having a vocation less factual than eidetic and being interested first of all in the contents of the concepts is still to be born.

But this is only one possible path: it belongs to the mathematicians of the twenty-first century to rethink the logical and philosophical foundations and, beyond that, the horizon of a science whose contours and possibilities are much less fixed than they seem. Far from the weight of dogmatisms of all kinds, let them therefore know how to remain open to the ever-renewed phenomenality of the world, since, in the end, and whatever the totalitarian aspirations of academic thought may be: “The quality of the researcher’s inventiveness and imagination lies in the quality of his attention to the voice of things. For the things of the Universe never tire of speaking about themselves and revealing themselves to those who care to hear” (A.Grothendieck).

# References

1. Alexandrov, P.S., In memory of Emmy Noether, in E. Noether, *Gesammelte Abhandlungen*, New York, Springer, 1983.
2. Aristotle, *Posterior Analytics*, transl. (French) J. Tricot, Paris, Vrin, 1979; (English) J. Barnes, Oxford, Oxford University Press, 1994.
3. Aristotle, *Metaphysics*, transl. (French) J. Tricot, Paris, Vrin, 1986; (English) J. Annas, *Metaphysics Books M and N*, Oxford, Oxford University Press, 1988.
4. Aristotle, *De Anima*, trans. (French) E. Barbotin, Paris, Les Belles Lettres, 1989; (English) Ch. Shields, Oxford, Oxford University Press, 2015.
5. Aristotle, *Poetics*, trans. (French) J. Hardy, Paris, Gallimard, 1996.
6. Arnold V. I., *Catastrophe Theory*, New-York, Springer, 1986.
7. Atiyah M. and Minio R., An interview with Michael Atiyah, *Mathematical Intelligencer*, vol. 6, 1984, p. 9–19.
8. Benoist J., *Autour de Husserl. L'ego et la raison*, Paris, Vrin, 1994.
9. Beaulieu L., *Bourbaki, une histoire du groupe de mathématiciens français et de ses travaux*, Université de Montréal, 1989.
10. Borel A., Twenty-five years with Nicolas Bourbaki, 1949-1973, *Notices of the American Mathematical Society*, vol. 45, no. 3, Mars 1998, 373–380.
11. Bourbaki, N., *Eléments de mathématiques*, Paris, Hermann, then Masson (1939–84).
12. Bourbaki, N., *L'architecture des mathématiques*, in *Les grands Courants de la Pensée mathématique*, Le Lionnais (ed.), Paris, Hermann, 1998.
13. Brunschvicg L., *Les Étapes de la philosophie mathématique*, Paris, Blanchard, 1981.
14. Cartan H., Nicolas Bourbaki and contemporary mathematics, *Mathematical Intelligencer*, 2, 1979-80, p. 175–180.
15. Cartier, P., *Vie et mort de Bourbaki. Notes sur l'histoire et la philosophie des mathématiques I*, Paris, IHES (1997).
16. Cartier, P., *Structures*, in D. Lecourt (ed.), *Dictionnaire d'histoire et philosophie des sciences*, Paris, Presses Universitaires de France, 1999.
17. Chevalley C. and Weil A., Hermann Weyl, *L'Enseignement Mathématique*, t. III, fasc. 3, 1957.
18. Chevalley C., Nicolas Bourbaki, *Collective Mathematician*, Interview by D. Guedj, *Mathematical Intelligencer*, 7, no. 2, 1985, p. 18–22.
19. Corry, L., *Modern Algebra and the Rise of Mathematical Structures*, Basel, Birkhäuser, 1996.
20. Dastur F., Heidegger et la question du temps, Paris, Presses Universitaires de France, 1990.
21. Dedekind R., *Gesammelte Werke*, 3. vol., Brunswick, 1930–1932.
22. Descartes R., *Oeuvres philosophiques*, Alquié, Paris, Bordas, 1988.
23. Descartes R., *Geométrie*, *Oeuvres*, ed. Adam et Tannery, vol. Paris, Vrin, 1996.

24. Diderot D. and D'Alembert J. *Le Rond, Encyclopédie, ou Dictionnaire raisonné des sciences, des arts et des métiers*, 17 vol. de texte et 11 vol. de planches, Paris, 1751–1772.
25. J. Dieudonné (ed.), *Abrégé d'histoire des mathématiques*, Paris, Hermann, 1978.
26. Dieudonné J., *Les grandes lignes de l'évolution des mathématiques*, IREM Paris-Nord, coll. Philosophie-mathématiques no. 1, 1980.
27. Dieudonné J., *The work of Bourbaki during the last thirty years*, *Notices of the Amer. Math. Soc.*, 29, 1982, p. 618–623.
28. Dieudonné J., *Panorama des mathématiques pures. Le choix bourbachique*. Rééd. J. Gabay, 2003.
29. Eilenberg S. and MacLane S., *Collected works*, Orlando, Academic Press, 1986.
30. Eilenberg S. and N. Steenrod, *Foundations of algebraic topology*, Princeton, Princeton University Press, 1952.
31. Frege, G., *Grundlagen der Arithmetik*, Breslau, Wilhelm Koebner, 1884; transl. J.L. Austin, *The Foundations of Arithmetic: A logico-mathematical enquiry into the concept of number*, Oxford, Blackwell, 1953.
32. Galileo, *Dialogo sopra i Due Massimi Sistemi del Mondo*, Florence, Giovanni Batista Landini, 1632; transl. S. Drake, *Dialogue Concerning the Two Chief World Systems*, Berkeley, University of California Press, 1967.
33. Gauss C.F., *Disquisitiones Arithmeticae*, 1801, transl. A. A. Clarke, Yale University Press, 1965.
34. Gauthier Y., *Logique et fondements des mathématiques*, Paris, Diderot, 1997.
35. Gonseth F., *La Géométrie et le problème de l'espace*, Neuchâtel, Éd. Du Griffon, 1945–1955.
36. Goursat, E., *Cours d'Analyse mathématique*, 3 vol., Gauthier-Villars, Paris, 1902–1913.
37. Grothendieck A., *Récoltes et semailles, réflexions sur un passé de mathématicien*, Montpellier, 1985.
38. Guérindon J. and Dieudonné J., *L'algèbre depuis 1840*, in J. Dieudonné (ed.), *Abrégé d'histoire des mathématiques*, Paris, Hermann, 1978.
39. Hegel, *Vernunft in der Geschichte*, 1837; transl. (French) K. Papaioannou, *La Raison dans l'histoire*, Paris, Plon, 1965.
40. Hegel, *Frühe Schriften*, Frankfurt, Suhrkamp, 1971.
41. Heidegger, M., *Was Heisst Denken?* M. Niemeyer, 1954; transl. F. D. Wieck and J. Glenn Gray, *What is Called Thinking?*, New York, Harper & Row, 1968.
42. Heidegger, M., *Überwindung der Metaphysik*, in *Vorträge und Aufsätze*, Pfullingen, 1954.
43. Heidegger, M., *Aus der Erfahrung des Denkens*, Günther Neske, Pfullingen, 1954.
44. Hermann, R., *Mathematics and Bourbaki*, *The Mathematical Intelligencer*, 8, 1986, p. 32–33.
45. Hilbert, D., *Grundlagen der Geometrie*, Leipzig, B.G. Teubner, 1899; transl. E.J. Townsend, *The Foundations of Geometry*, La Salle, Open Court Publishing, 1950.
46. Hilbert, D., *Lecture at the International Congress of Mathematicians*, Paris, 1900.
47. Houzel, C., *The birth of non-euclidean geometry*, in L. Boi et al. (eds.), *1830–1930, A Century of geometry*, Berlin, Springer, 1992.
48. Husserl, E., *Psychologische Studien zur elementaren Logik*. *Philosophische Monatshefte* 30, 159–191, 1894.
49. Husserl, E., *Cartesianische Meditationen und Pariser Vorträge*, Den Haag, Nijhoff, 1950; transl. D. Cairns, *Cartesian Meditations*, Dordrecht, Kluwer, 1988.
50. Husserl, E., *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, *Husserliana* (Hua) III, Hua IV, Hua V, Den Haag, Nijhoff, 1952; transl. F. Kersten; T. E. Klein and W. E. Pohl; R. Rojcewicz and A. Schuwer, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy*, The Hague, Nijhoff and Dordrecht, Kluwer, 1982, 1989, 1980.
51. Husserl, E., *Die Krisis der europäischen Wissenschaften und die transzendente Phänomenologie*, Hua VI, Den Haag, Nijhoff, 1954; transl. D. Carr, *The Crisis of European Sciences and Transcendental Phenomenology*, Evanston, Northwestern University Press, 1970.
52. Husserl, E., *Logische Untersuchungen*, Hua XVIII, Hua XIX/1, Hua XIX/2, Den Haag, Nijhoff, 1975–84; transl. J. N. Findlay, *Logical Investigations*, London, Routledge 1973.

53. Israel, G., *La mathématisation du réel*, Paris, Seuil, 1996.
54. Jacobi, C.G.J., *Gesammelte Werke*, Berlin, 1881.
55. Janicaud, D., *La puissance du rationnel*, Paris, Gallimard, 1985.
56. Kant, E., *Kritik der reinen Vernunft*, 1781–87; transl. P. Guyer and A. Wood (eds), *Critique of Pure Reason*, Cambridge: Cambridge University Press, 1998.
57. Kant, E., *Logik in Gesammelte Schriften*, Berlin: G. Reimer then W. De Gruyter, 1910–1983, vol. IX; transl. J. Young (ed.), *Lectures on Logic*, Cambridge: Cambridge University Press, 1992.
58. Kant, E., *Dissertatio* (1770), *De mundi sensibilis atque intelligibilis forma et principiis*, transl. D. Walford and R. Meerbote (eds.), *Inaugural Dissertation*, in *Theoretical Philosophy*, 1755–1770, Cambridge: Cambridge University Press, 1992.
59. Klein, F., *Vergleichende Betrachtungen über neuere geometrische Forschungen* [A comparative review of recent researches in geometry; known as the Erlangen program], *Mathematische Annalen*, 43 (1893) 63–100, transl. (French), *Le programme d'Erlangen*, Paris, J. Gabay, 1991.
60. Kuhn, T., *The Structure of Scientific Revolutions*, University of Chicago Press, 1962.
61. Lacan, J., *Écrits*, Paris, Le Seuil, 1966.
62. Lakatos, I., *Proofs and Refutations*, Cambridge: Cambridge University Press, 1976.
63. Lakatos, I., *Histoire et méthodologie des sciences*, transl. (French) C. Malamoud and J.-F. Spitz, Paris, Presses Universitaires de France, 1994.
64. Lawvere, F. W., *Categories of space and of quantity*, in J. Echeverria (ed.), *Philosophical, epistemological and historical explorations*, Berlin, Walter de Gruyter, 1992.
65. Leibniz, *Mathematische Schriften*, Hildesheim, G. Olms, 1971.
66. F. Le Lionnais (ed.), *Les grands Courants de la Pensée mathématique*, reed. Paris, Hermann, 1998.
67. Levi-Strauss, C., *Anthropologie structurale*, Paris, Plon, 1958.
68. Lukács, G., *Der junge Hegel*, Zürich, Europa-Verlag, 1948.
69. Mac Lane, S., *Categories for the Working Mathematician*, Berlin, Springer (1971).
70. Mandelbrot, B., *Les Objets fractals: forme, hasard, dimension*, Paris, Flammarion, 1975.
71. Mandelbrot, S., *Pourquoi je fais des mathématiques*, *Revue de Métaphysique et de Morale*, 57, cah. 4, 1952, p. 429–442.
72. Mathias, A., *The ignorance of Bourbaki*, *The Mathematical Intelligencer*, 14, no. 3, 1992, p. 4–13.
73. Peckhaus, V. *Hilbert's programme and critical philosophy*, Göttingen, Vandenhoeck and Ruprecht, 1990.
74. Reid, C., *Hilbert*, Berlin, Springer, 1970.
75. Russell, B., *Letter to Frege* (1902), in Van Heijenoort J., *From Frege to Gödel*, Cambridge (Mass.)/London, Harvard University Press, 1967.
76. Samuel, P., *Géométrie projective*, Paris, Presses Universitaires de France, 1986.
77. Schwartz, L., *Un Mathématicien aux prises avec le siècle*, Paris, Odile Jacob, 1997.
78. Singh, S., *Fermat's last theorem*. Fourth Estate, 1997.
79. Thom, R., *Stabilité structurelle et Morphogénèse*, Interéditions, Paris, 1972.
80. Thom, R., *Paraboles et Catastrophes*, Paris, Flammarion, 1983.
81. Thom, R., *Prédire n'est pas expliquer*, Paris, Eshel, 1991.
82. van der Waerden, B., *Moderne Algebra*. Teil I, Springer-Verlag, 1930, Teil II, 1931.
83. Vuillemin J., *Mathématiques et métaphysique chez Descartes*, Paris, Presses Universitaires de France, 1960.
84. Weil, A., *Oeuvres Scientifiques*, New York, Springer, 1979.
85. Weil, A., *The Apprenticeship of a Mathematician*, Basel, Birkhäuser, 1992.
86. Weyl, H., *Die Idee der Riemannschen Fläche*, Leipzig, 1913.
87. Weyl, H., *David Hilbert and his Mathematical Work*, *Bulletin of the American Mathematical Society*, 50, 1944, p. 612–654.
88. Weyl, H., *A half-century of mathematics*, *The American Mathematical Monthly*, 58, 1951, p. 523–553.
89. Weyl, H., *Symmetry*, Princeton University Press (1952).

90. Weyl, H. *Symétrie et mathématiques modernes* (French transl. of *Symmetry*), Paris, Flammarion, 1964.
91. Zerner, M., *Analyse non standard*, in D. Lecourt (ed.), *Dictionnaire d'histoire et philosophie des sciences*, Paris, Presses Universitaires de France, 1999.